GRADATIONS

IN

ALGEBRA,

IN WHICH

THE FIRST PRINCIPLES OF ANALYSIS

ARE

INDUCTIVELY EXPLAINED.

ILLUSTRATED BY

COPIOUS EXERCISES,

AND MADE SUITABLE FOR PRIMARY SCHOOLS.

BY

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LITERAL ALGEBRA.

PREFACE.

The object of the author, in composing this treatise, was to form an easy introduction to the first principles of algebraical reasoning; and also to embrace, in the same course, a popular exposition of the most important elements of arithmetic. And he believes that he has been enabled to combine the rudiments of both, in such a manner as to make the operations of one illustrate the principles of the other.

In order that this method of treating the subject might preserve its chief advantage, especially in the initiatory course of the study; the work has been divided into two parts—*Numeral Algebra* and *Literal Algebra*.

In Numeral Algebra I have treated of the several arithmetical operations; first making them intelligible to very young pupils, and then exhibiting them under the algebraical notation. By this means, as every lesson in algebra is immediately preceded by corresponding numerical exercises, the transition from one to the other has been made so trifling, that the pupil will feel at each step that he has met with nothing more than what he has already made himself familiar with, in a different dress. Besides, as algebraical operations require the exercise of abstraction in a greater degree than the pupil is supposed to be accustomed to, I have taken care that the exercise on each of the fundamental rules, shall be followed by a selection of problems to be solved by equations.

As mathematical questions of this kind are always pleasing to young pupils, this arrangement will serve to impart an interest to the study at the commencement, and also to preserve a taste for it through the whole work.

Under the head of Literal Algebra, I have repeated, in

a more strictly *algebraical form*, the principles which have been explained in the preceding part of the work; and have shown some of their uses by applying them in the deduction and demonstration of several abstruse operations on numbers.

But the great peculiarity of the book is, that it habituates the *speech* and the *ear* to mathematical language. In any study, it is necessary for beginners to receive such a course of training as will imprint upon their minds each new idea, as soon as it is apprehended. Learners in the mathematics especially, are accustomed to forget soon, both the names and the use of the signs; and also the arrangement of the several steps in the solution of their problems. On this account I have required the pupil *always to repeat verbally* the operation that he has performed; taking care to omit no part of the work that would hinder an auditor from understanding the reason for the several steps, and consenting to the just conclusions of the answer which has been obtained. It has been found by experience that this simple device enables the young pupil to acquire the science very easily; and while it impresses his lessons indelibly upon his memory, it also developes his genius, rectifies his inventive faculties, and imparts, as it were, a mathematical form to his mind; so much so, that he is generally capable of pursuing the subject afterwards by himself.

In order to accomplish this end more perfectly, I have swelled the number of examples beyond the ordinary limits. These should be thoroughly mastered as the pupil proceeds. There must be no smattering in the beginning of a science if the learner is to continue the study. The author has found by long experience, that a book is sooner finished when each part has been made familiar to the mind, than when it has been superficially attended to.

With regard to the arrangement of the several divisions, I have been careful to introduce first those principles that will be the most easily apprehended; and afterwards such others as would most naturally arise from the former if the study were entirely new. This method appears to be the best adapted for teaching the rudiments of a science; although in a succeeding text book, it is necessary that the arrangement of the several parts should be more systematic. On this account the advanced scholar must not be surprised to find in the middle of the book, what he has been accustomed to see near the beginning of other treatises. However, so much regard to a regularity of arrangement has been attended to, that the pupil will be assisted by the associations of method, both to understand and to remember.

As the author wishes to bring the study of Algebra within the reach of common schools, he has endeavored to prepare this work, so that it may be studied by pupils who are not already adepts in arithmetic. And it is believed that such learners will not fail of obtaining, by a perusal of it, a full understanding of vulgar fractions, roots and powers, proportion, progression, and other numerical operations that are generally embraced in arithmetical treatises.

ADVERTISEMENT.

The foregoing is the Preface of the author's "Inductive Treatise on the Elementary Principles of Algebra." The first 146 pages of that book have been published in this form, in order to afford a cheap manual for those classes that do not wish to study beyond *Simple Equations*. In the present state of education, *so* much of Algebra should be studied by every pupil in our common schools.

R. W. G.

ERRATA.

Before the pupil begins to perform the sums in this work, he is requested to make the following corrections:

Page 17 line 15 erase *comma* in 130.
" 43 " 11 instead of *itself*, read *the multiplier*.
" 26 " 2 for $36x$, read $35x$.
" 31 " 6 for 173, read $17z$.
" 35 " 6 for 76, read 66.
" 38 " in §54, for 32, read 37.
" 41 " 9, for 24 read 20.
" 45 example 5, for 181, read 367.
" " " 9, for $\frac{9}{17}$, read $\frac{11}{17}$.
" 48 line 1, for 30 read 80.
" 55 " 10 insert $\frac{2}{3}$.
" 65 example 13, for $14x-63$, read $14x+63$.
" 66 " 5, for 273, read 213.
" 67 " 5, for 1000, read 2280.
" 70 " 14, for $7-\frac{2x}{3}$, read $\frac{7-2x}{3}$.
" 74 " 4, for $-\frac{2x}{3}$, read $+\frac{2x}{3}$.
" 75 " 8, for $3+\frac{2x}{9}$, read $\frac{3+2x}{9}$.
" " " 10, for $6+2y$, read $3+2y$.
" 77 line 24, for 23, read 33.
" 86 example 6, for $-3y$ and $-6y$, read $+3y$ and $+6y$.
" 90 example 8, in ans. for $4s$, read $5s$.
" 91 " 12, for 35, read 31.
" 93 line 4, for $\frac{1}{6}$, read $\frac{1}{5}$.
" 97 " 7, for 10, read 16.
" 112 example 1, for $4x$, read $4ax$.

PART I.

NUMERAL ALGEBRA.

PRELIMINARY REMARKS.

§1. Algebra is a method of computation, in which we find the value of an unknown quantity, by using, in our calculations, *that quantity itself.*

Example 1. A friend has told me, that in both of his vest pockets he has 28 cents; but in the right pocket there are 8 cents more than in the left.—I will use the following method of finding how many cents there are in each pocket.

By the statement, the right pocket contains as much as the left pocket and 8 cents besides.

Therefore, instead of saying, 28 cents is as much as in the left pocket and also in the right pocket; we may say 28 cents is as much as in the left pocket, and again in the left pocket, and 8 cents besides.

Hence, 28 cents is as much as there is in the left pocket, *twice*, and 8 cents besides. And, as this is the case,

27 cents is as much as twice the left pocket and 7 cents.

26 cents is as much as twice the left pocket and 6 cents.

And so on, making the cents fewer and fewer; at last,

20 cents is as much as twice the left pocket.

Therefore 10 cents is as much as once the left pocket; that is, there is 10 cents in the left pocket. Of course there is 18 cents in the right pocket.

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Example 2. A brother told his sister, that he was 4 years older than she; and that his age and her age put together, made 18 years. The age of each may be found as follows.

By the statement, *his* age is the same as *her* age and 4 years besides. Therefore, instead of saying 18 years is as much as *her* age and *his* age put together; we can say

18 years is as much as *her* age, and also *her* age and 4 years more.

But *her* age, and *her* age, is the same as *twice* her age.

Therefore, 18 *years* is as much as *twice her age* and 4 *years* more. And by taking away the 4 years,

14 *years* is as much as *twice her age.*

Therefore 7 years is once *her* age; and as he is 4 years older, his age is 11 years.

§2. The method in which the two preceding sums were performed, was in use for many years. But in course of time, mathematicians began to abbreviate the *words* that were made use of in their operations. We will proceed to show a few of these abbreviations.

§3. About the year 1554, Stifelius, a German, introduced the sign $+$ for the words *added to;* and called it *plus.* Ever since that time, $+$ has been used *to signify that the quantity after it, has been added to the quantity before it.* Thus, $2+6$, is read 2 *plus* 6; and is the same as, 2 *with* 6 *added to it.*

Example 3. Divide \$1000 among A, B, and C; so that B shall have \$72 more than A, and C shall have \$172 more than A.

By the statement, B's share is equal to A's and \$72 besides. We may write it thus, *A's*+\$72.

C's share is equal to A's and \$172 besides. We may write his share thus, *A's*+\$172.

Then all three shares put together, will be written, *A's,*

and $A's+\$72$, and $A's+\$172$; or, $A's+A's+\$72+A's+\$172$.

Therefore, $A's+A's+\$72+A's+\$172$, is equal to \$1000.

Putting together the \$72 and \$172,

$A's+A's+A's+\$244$, is equal to \$1000.

Then, 3 times $A's+\$244$, is equal to \$1000.

Taking away \$244, 3 times $A's$, is equal to \$756.

Of course, once $A's$ share is equal to $\frac{1}{3}$ of \$756; which is \$252.

$B's$ share is $\$252+\$72$; which is \$324.

$C's$ share is $\$252+\$172$; which is \$424.

Proof, \$1000.

§4. The student will see that in the operation of the last sum, instead of writing *A's share*, we have written simply *A's*. This method of abbreviation was used for for a long time before Stifelius wrote. But he simplified the operation, by employing only some capital letter for every unknown number.

Example 4. My purse and money together, are worth 40 shillings; and the money is 7 times the worth of the purse. How much money is in the purse?

Now instead of writing *the worth of the purse*, we will represent it by the capital P; which may well stand for *Purse*. Then the *money* will be as much as 7 *times P*.

Therefore, $P+7$ times P, is as much as 40s. Putting the $P's$ together, 8 times P is equal to 40s. And of course,

Once P *is* $\frac{1}{8}$ of 40s; which is 5s.

The money is 7 times 5s; which is 35s.

Proof, 40s.

§5. About the year 1550, John Scheubelius, a German,

introduced the following practice. Instead of writing 2 *times* A, or 3 *times* A, or 4 *times* B, &c.; he wrote 2 A, 3 A, 4 B, &c.

Example 5. Two men, A and B, owe me \$270; and B's debt is twice as much as A's. How much does each of them owe me?

We will represent A's debt by A; and then B's debt will be twice as much, or 2 A.

But both, when put together, are equal to \$270.

Therefore, $A+2A$, is equal to \$270.

Putting the A's together, 3A, is equal to \$270.

Then A's debt is $\frac{1}{3}$ of \$270; which is \$90.

B's debt is twice as much; which is \$180.

Proof, \$270.

§6. In 1557, Dr. Recorde, an Englishman, introduced the sign $=$, which we call EQUALS. It is used instead of the expression, *is equal to*, or, *is as much as*. Thus, $2+6=8$, is read, 2 *plus* 6, *equals* 8.

Example 6. A's age is double of B's; and B's age is three times C's; and the sum of all their ages is 70 years. What is the age of each?

Let us represent C's age by the capital C. Then B's age will be three times as much; that is 3C. And A's age will be twice that, which is 6C. Then all of them will be, $C+3C+6C$. Therefore

$$C+3C+6C=70.$$

Putting the C's together, $10C=70$.

Dividing by 10, $C=7$.

C's age=7 years.

B's is three times as much, or 21 years.

A's is twice *B's;* or 42 years.

Proof, 70

§7. We very often wish to state that we have made a quantity to be less. This was formerly done by using the word MINUS. Thus, if I wished to say that your age is 6 years less than mine; I may say your age is equal to mine when 6 years are subtracted from mine; or, in fewer words, your age is equal to my age *minus* 6 years.

§8 Stifelius introduced the sign — for *minus;* so that 20—6, is read 20 *minus* 6; and signifies, 20 with 6 *subtracted* from it. Thus, $20-6=14$.

Example 7. At a certain election, 548 persons voted; and the successful candidate had a majority of 130 votes. How many voted for each?

Let us represent the number of votes received by the successful candidate, by the letter *A*. Then, as the unsuccessful candidate received 1,30 votes less, his number may be represented by *A*—130.

Then $A,+A-130,=548$

Putting the A's together, $2A-130=548$ which means, 2A, after 130 is subtracted from it, is equal to 548. Then, 2A, before 130 was subtracted from it, was equal to 130 more than it is now. That is, it was equal to 548 and 130 more. Therefore we say adding the 130, $2A=678$.

Dividing by 2, $A=339$.

The successful candidate had 339.

The unsuccessful one, 130 less=209.

Proof 548.

§9. *Des Cartes*,* a Frenchman, who wrote about 1637, used the *last letters of the alphabet;* namely, x, y, z, u, &c. to denote the *unknown quantities.* And this is now the practice of all mathematicians.

* Pronounced *Da-Cart'*.

Example 8. It is required to divide \$300 among *A*, *B*, and *C;* so that *A* may have twice as much as *B*, and C may have as much as *A* and *B* together.

Let us represent B's share by x. Then A's share will be $2x$; and C will have as much as both put together, which is $3x$. Then,

$$x+2x+3x=300.$$

Putting the x's together, $6x=300.$

Dividing by 6, $x=50.$

Therefore, B's share is \$50.
A's is twice as much, or \$100.
C's=as much as both, or \$150.

Proof, \$300.

§10. As we sometimes wish to speak of particular parts of our calculations, mathematicians have given the name *term* to any quantity that is separated from others by one of the SIGNS + or —. Thus, in the last Example, and first line of the operation, the first x is the first term, the $2x$ is the next term, and the $3x$ is the next term; and the 300 is the last term.

§11. When a figure is put before a *letter* to denote how many times we take the quantity which that letter stands for, the *figure* is called a *co-efficient.** Thus, in the term $2x$, 2 is a co-efficient of x; in the term $3A$, 3 is a co-efficient of A.

§12. It must also be understood, that a letter *without* any number before it, has 1 for its co-efficient. Thus, x represents $1x$; $a=1a$; &c. The 1 is omitted because it is plainly to be understood.

* This name was given by Franciscus Vieta, about 1573.

§13. Any number, or letter, or any thing else, used to denote a quantity, when it is not united to any other quantity by either the sign + or —, is called a *simple quantity*. Thus, x is a simple quantity; $2x$ is a simple quantity; 12 is a simple quantity; &c.

ADDITION AND SUBTRACTION OF SIMPLE QUANTITIES.

§14. In Algebra, *simple quantities are added by writing them down, one after another; being careful to put the sign + between them.* Thus we add 9 to x, by writing them $9+x$, or, $x+9$.

☞ The pupil must understand that x stands for some number; but it is often the case that we do not know what that number is.

§15. It will readily be seen, that it is of no consequence which quantity is put first; for $9+7$ is the same amount as $7+9$.

§16. *One simple quantity is subtracted from another simple quantity, by writing down both quantities, one after another, and putting the sign — before the quantity which we subtract.*

Thus, we subtract 4 from x, by writing them $x - 4$.

§17. When two simple quantities have been added, or one simple quantity subtracted from another, they then will consist of more than one term.

§18. Quantities that consist of more than one term, are

called *compound quantities.* Thus, $a+b$, is a compound quantity. So is $a-b$, and $x+7$, and $x-7$, &c.

§19. It sometimes happens, that in those compound quantities which are made by adding or subtracting, there are two or more terms of the same kind; such as $x+2x$, $4a-2a$, $5x-3x+x$. Such quantities are called *like* quantities.

§20. When in any compound quantity, there are two or more terms of the same kind, they may be *united* by performing the operation which is expressed by the sign. Thus, $x+2x$ is united into $3x$. $4a-2a$ is united into $2a$. $5x-3x+x$ is united into $3x$.

§21. Numeral quantities* may be united in the same manner. Thus, $4+3$ may be united into 7. $9-4$ may be united into 5. $6-2+5$ may be united into 9.

EXAMPLES.

Unite, as much as possible, the terms in each of the following compound quantities.

1. $a+b+8-4+2a+6+3b-b+3a$. Take one letter and go through the whole quantity with that first; and then take another letter and do the same. And then another, &c.

Ans. Uniting the a's, they equal $+6a$; uniting the b's, they equal $+3b$; uniting the numeral quantities, they equal $+10$. Therefore the answer is, $6a+3b+10$.

2. $2a+b+4a+2b+8+6-3a-b-4$.

3. $3x+y+z+3x+3y+z+4x-2y+z$.

4. $18-9+5+8a+4x+7x+5a-6x-7a$.

* Numeral quantities are expressed by *figures*; and literal quantities, by *letters*.

5. $4y+7z+9a—5a—4z—y+2z+3a+2y$.

6. $7a—a—a+5b+4z+a—2z+a—z+4a$.

7. $8—4+6+5—2+8+x+3x—2x+4x$.

§22. When any simple quantity begins, with the sign $+$, it is called a *positive quantity ;* as $+a, +3a$.

§23. When any simple quantity begins with the sign —, it is called a *negative quantity ;* as, $—6$, $—5a$.

§24. In algebra, the *perfect* representation of any simple quantity requires both the specified sum, and either the sign $+$, or the sign — ; as, $+5$, $—40$, $+x$, $—3x$.

§25. But, when a *positive* quantity stands by itself, or when it is the first term of a compound quantity ; the sign that belongs to it, is generally omitted on paper, and also in our reading ; as, x, 2, $a+b$, $x—y$.

§26. Therefore, when a simple quantity, or the first term of a compound quantity, does not begin with a sign, we say that the sign $+$ is understood. That is, we think of the quantity the same as if $+$ was before it.

§27. In reading compound quantities, the pupil must be careful to join the sign to the term that is *immediately after it.* Thus, the first example under §21, must be read *a ; plus b ; plus* 8 ; *minus* 4 ; *plus* 2*a* ; &c.

We are now ready for the following

GENERAL RULE FOR UNITING TERMS.

§28. Select one kind of like quantites, and add into one sum all the *positive* co-efficients that belong to them. Then add into another sum all the *negative* co-efficients that belong to them. Then subtract the less sum from the greater; and prefix the *sign of the greater* to the difference, annexing the common letter.

NOTE.—It sometimes happens that the negative quantity is greater than the positive quantity. In such cases, the difference will have the sign —.

Examples. Unite the terms in the following.

1. $3a+4b+2b-3c+3c-b+a$.

Ans. 1st, $4a+6b-b+3c-3c$.

Now $+6b-b=+5b$; and $+3c-3c$ balance each other, so as to be equal to 0. The final answer is $4a+5b$.

2. $a+b+3a-c+4a-3c+b$. Ans. $8a+2b-4c$.

3. $5x+5y+3x-2y$. Ans. $8x+3y$.

4. $b+a+3a-5b+4x-a+3a-x$. Ans. $6a-4b+3x$.

5. $4a-5b-10+2y-x+4$. Ans. $4a-5b+2y-x-6$.

6. $2x+3y-3x+5y-x-z$. Ans. $-2x+8y-z$.

7. $x+z+x-z+x+3y$. Ans. $3x+3y$.

8. $3a-5-4b+6-2a-3+6b$. Ans. $a+2b-2$.

9. $4x+4+5y+6-2+3x-2y$. Ans. $7x+3y+8$.

10. $4c+3a-b-3a+c-2a-b$. Ans. $5c-2a-2b$.

11. $2b-c+3a-b+4c+2d-b-5d$.

12. $a-b+c+d+3a+b-2c-4a+10+a$.

13. $a+b+3b+x-7+4x-6+3a-y+2$.

14. $x-4+10-7x+a-b-10+2a-1$.

15. $3a-b+4c+2x-c+4a+5c+7b$.

16. $10-a+6-b-x+7+3b+2a-40$.

17. $8a-16-z+91+2y-87-3z+14$.

18. $29+46+27+y-32-43+y$.

19. $73+4x-36-3x-41+7x-y+2x$.

20. $a-b-a-b-67-42+7a+3b$.

MULTIPLICATION AND DIVISION OF SIMPLE QUANTITIES.

§29. Were we to add a to itself four times, we should write the sum thus, $a+a+a+a$; which, when united, becomes $4a$. Whence we see that any literal quantity is multiplied by a number, by putting that number before it as a *co-efficient;* as, 7 times x is $7x$. 6 times a is $6a$.

The pupil may now multiply x by each of the numbers from 2 to 20.

§30. If the quantity to be multiplied has already a co-efficient, that co-efficient *only* is to be multiplied. Thus, $3x$ taken four times is $3x+3x+3x+3x$, which when united $=12x$. The *co-efficient* is multiplied by 4. Thus, 4 times $3x=12x$.

The pupil may multiply $2x$ by each number from 2 to 12.

He may then multiply $3x$ by the same numbers; and then $4x$ by the same.

§31. It is evident that if 2 times $3x$ is $6x$, then one half of $6x$ is $3x$. Whence we learn that a quantity with a numeral co-efficient, may be divided, by merely dividing that co-efficient. Thus, $8x$ divided by $2=4x$. $12x$ divided by $4=3x$, &c.

§32. In 1661, Rev. William Oughtred of England, published a work, in which he introduced the sign $\times$ to represent multiplication. Thus, $4\times3=12$, is read 4 *multiplied* by 3 equals 12; or, 4 *into* 3 *equals* 12.

§33. In 1668, Mr. Brancker invented the sign $\div$ for division. This sign is always put *before the divisor;* as $20\div4=5$; read 20 divided by 4 equals 5; or, 20 *by* 4, *equals* 5.

SIMPLE EQUATIONS.

§34. The most general application of algebra, is that which investigates the values of unknown quantities by means of *equations.*

§35. An equation is an expression which declares one quantity to be equal to another quantity, by means of the sign $=$ being placed between them.

Thus, $5+3=8$, is an equation, denoting that 5 with 3 added to it, equals 8. Also, $4-1=3$, and $3+2-1=4$, and $8-2=5+1$, are equations, each denoting that the quantity on one side of the $=$, is equal to that on the other side.

§36. The whole quantity on the left of $=$ is called the *first member of the equation;* and all on the right of $=$ is called the *last member of the equation.*

In order to be a member of the equation, it is of no importance whether the quantity is simple or compound. Thus, in the equation, $x=4+a-b-18$, x is the first member, and $4+a-b-18$ is the last member. And in this case, the first member represents just as great a quantity as the last.

§37. In order that an equation may be such that we can find the value of an unknown quantity by it, it must contain some quantity that is already known. And then, we find the value of the *unknown* quantity, by making that stand by itself on one side of $=$, and all the known quantities on the other side; taking care to change them in such a manner as not to destroy the equation.

§38. The operation of managing an equation, so as to bring the unknown quantity to stand equal to a known quantity, is called *solving or reducing the equation.*

EQUATIONS.—SECTION 1.

Equations which are solved by merely uniting terms.

In each of the following equations, the object is to find the value of x.

Example 1. $x+2x=45-15$.

Uniting terms, $3x=30$.

Now, as we have found that *three* x's$=30$, it is evident that *one* x will be one third of 30. Therefore, dividing by 3, $x=10$.

2. $8x-4x-x=7+26+51-15$

Uniting terms, $3x=69$

Dividing by 3, $x=23$.

3. $10x-5x+4x=56+75+32-1$

Uniting terms, $9x=162$

Dividing by 9, $x=18$.

4. $x+2x+3x+4x=12+35+74-11$

Uniting terms, $10x=110$

Dividing by 10, $x=11$.

5. $8x-3x+2x=46+54+37-4$. Ans. $x=19$.

6. $4x-3x+4x=29-36+48+14$. Ans. $x=11$.

7. $6x-8x+14x=12+36+14+22$. Ans. $x=7$.

8. $5x+4x+3x=49+14+22+11$. Ans. $x=8$.

9. $7x+x=14-22-11+41-6$. Ans. $x=2$.

10. $4x-2x=96-7+8-15-10$. Ans. $x=36$.

11. $5x-x=2+3-15-10+72$. Ans. $x=13$.

12. $6x=7+4+72-51-16-10$. Ans. $x=1$.

13. $8x-7x+5x-4x+3x=27-12$. Ans. $x=3$.

14. $5x-4x+2x-3x+x=39-13$. Ans. $x=26$.

15. $17x-4x-3x-5x-x=57-32$. Ans. $x=5$.

16. $14x-36x+29x+47x+x=504$. Ans. $x=9$.

PROBLEMS.

§39. An algebraic problem is a proposition which requires the discovery or demonstration of some unknown truth.

§40. In the solution of problems, the first thing to be done, is to make a statement of the conditions, in algebraic language, in the same manner as if the answer were already found, and you were required to see if it is right. In order to do this, it is customary to represent the unknown quantity by x, y, or some other final letters of the alphabet.

§41. When the question has been fairly stated, it will be found that some condition has been represented in two ways; one having the *unknown* quantity in it, and the other having a *known* quantity. These two expressions must be put together, so as to form an *equation*. And then, by reducing the equation, the required result will be found.

1. The sum of \$660 was subscribed for a certain purpose, by two persons, A and B; of which, B gave twice as much as A. What did each of them subscribe?

Stating the question, A gave x dollars.
B gave $2x$ dollars.
Both of them gave $x+2x$ dollars.
But both gave 660 dollars.

Therefore, putting the question into an equation,

$$x+2x=660$$

Uniting terms, $3x=660$

Dividing by 3, $x=220$ A's share.

$2x=440$ B's share.

660 proof.

2. Three persons in partnership, put into the stock

\$4800; of which, A put in a certain sum, B twice as much, and C as much as A and B both. What did each man put in?

Stating the question, x=A's share.

$2x$=B's share.

$x+2x$=C's share.

$x+2x+x+2x$= the whole.

4800= the whole.

Therefore, forming the equation,

$$x+2x+x+2x=4800$$

Uniting terms, $6x=4800$

Dividing by 6, $x=800$ A's share.

$2x=1600$ B's share.

$800+1600=2400$ C's share.

4800 proof.

☞ In all the succeeding problems, the learner should prove his answers.

3. A person told his friend that he gave 108 dollars for his horse and saddle; and that the horse cost 8 times as much as the saddle. What was the cost of each?

Stating the question, x= price of the saddle.

$8x$= " horse.

$x+8x$= " both.

108= " do.

Forming the equation, $x+8x=108$

Uniting terms, $9x=108$

Dividing by 9, $x=12$= price of saddle.

It is advisable for the pupil while performing his sums, to write them on his slate in a manner similar to the three questions above; beginning the statement by making x the answer to the *question*. And in recitation, the whole of it is to be recited.

4. A father once said, that his age was six times that of

his son; and that both of their ages put together, would amount to 49 years. What was the age of each? Ans. Son's age 7 years; father's 42.

5. A farmer said that he had four times as many cows as horses, and five times as many sheep as cows; and that the number of all of them was 100. How many had he of each sort. Ans. 4 horses; 16 cows; and 80 sheep.

6. A boy told his sister that he had ten times as many chestnuts as apples, and six times as many walnuts as chestnuts. How many had he of each sort, supposing there were 639 in all. Ans. 9 apples; 90 chestnuts; and 540 walnuts.

7. A school girl said that she had 120 pins and needles; and that she had 7 times as many pins as needles. How many had she of each sort? Ans. 15 needles, and 105 pins.

8. A teacher said that her school consisted of 64 scholars; and that there were three times as many in arithmetic as in algebra, and four times as many in grammar as in arithmetic. How many were there in each study? Ans. 4 in algebra; 12 in arithmetic; and 48 in grammar.

9. Two men, who are 560 miles apart, start to meet each other. One goes 30, and the other goes 40 miles a day. In how many days will they meet? Ans. 8 days.

Remarks. Each will travel x days. The first will go x times 30 miles, and the second will go x times 40 miles; and both together will go the whole distance. It is also evident that x times 30 is the same as 30 times x; &c.

10. A teacher had four arithmeticians who performed 80 sums in a day. The second did as many as the first, the third twice as many, and the fourth as much as all the

other three. How many did each perform? Ans. The first and second, each 10; the third, 20; and the fourth, 40

11. A person said that he was $450 in debt. That he owed A a certain sum, B twice as much, and C twice as much as to A and B. How much did he owe each? Ans. To A $50, to B $100; and to C $300.

12. A person said that he was owing to A a certain sum; to B four times as much; and to C eight times as much; and to D six times as much; so that $570 dollars would make him even with the world. What was his debt to A? Ans. $30.

13. A man bought 3 sheep and 2 cows for $60. For each cow, he gave 6 times as much as for a sheep. How much did he give for each?

Remarks. If $x=$ price of a sheep, all the sheep will cost three times as much, or $3x$. In the same manner both cows will cost twice as much as one cow. One cow will cost $6x$, and 2 cows will cost $12x$. Ans. $4, price of a sheep; and $24 price of a cow.

14. A gentleman hired 3 men and 2 boys for one day. He gave five times as much to a man as he gave to a boy; and for all of them he gave $6.80. What was the wages of each? Ans. A boy's wages was 40 cents, and a man's wages, $2.

15. A boy bought some oranges and some lemons for 54 cents. There was an equal number of each sort, but the price of an orange was twice the price of a lemon. How much money did he spend for each sort? Ans. 18 cents for lemons; and 36 cents for oranges.

16. A boy bought some apples, some pears, and some peaches, an equal number of each sort, for 72 cents. The price of a pear was twice that of an apple, and the price

of a peach was 3 times that of an apple. How much money did he give for each kind? Ans. 12 cents for apples; 24 cents for pears; 36 cents for peaches.

17. A farmer hired three labourers for \$50.00; giving to the first \$2.00 a-day, to the second \$1.50, and to the third \$1.00. The second worked three times as many days as the first; and the third twice as many days as the second. How many days did each work? Ans. The first, 4; second, 12; and third, 24 days.

18. A gentleman bought some tea, coffee, and sugar, for \$7.04; giving twice as much a pound for coffee as for sugar, and five times as much for tea as for coffee; and there were 20 pounds of sugar, 12 pounds of coffee, and 2 pounds of tea. What was the price of each? Ans. 11 cents for sugar; 22 cents for coffee; and 110 cents for tea.

ADDITION OF COMPOUND QUANTITIES.

§42. When two or more expressions that consist of several terms, are to be added together, the operation is represented by connecting them with one another by means of the sign $+$. Thus, $a-x$ is added to $y+7$ in the following manner:

$$a-x+y+7, \text{ or } y+7+a-x.$$

For to $a-x$ we add first y, and then 7; or to $y+7$ we add a, and then because we ought to have added $a-x$, we see that we have added x too much, and therefore subtract it.

§43. The above example shows that it is of no consequence in what order we write the terms. Their place may be changed at pleasure, provided their signs be preserved.

§44. After compound quantities have been added, their terms may be united according to the rule §28.

EXAMPLES.

1. Add the following compound quantities. $2a-8x$, $x-3a$, $-4a-2x$, $4x-a$. Ans. $-6a-5x$.

2. Add $2-x+4y$, $3+3x-y$, $-30-x-2y$, and $1-2x+3y-10z$ together. Ans. $24-x+4y-10z$.

3. Add $3x+5y-6z$, $-2x-8y-9z$, $20x+2y-3z$ and $x-y+z-4$ together. Ans. $22x-2y-173-4$.

4. Add $3-2y+z$, $4y-2z+5$, $2-z-y$, and $2z-y-10$. Ans. Nothing.

5. Add $7x-6y-5z-8-g$
$3-g-3y-x$
$7g-1-3z+y-x$
$3z-1-g+3y-2x$
$x+8y-5z+9+g$. Ans. $4x+5g+3y+2$.

6. Add $3b-a-c-115d+6e-5y$
$3a+27e-d-3c-2b$
$3e-7y-8c+5b$
$17c-6b-7a+9d-5e+11y$
$-2d-6e-5c-9y-3a+x$.
Ans. $37e-8a-109d-10y+x$.

7. Add $4-3a+x-7+4y$
$8+6y-3x+6a+32$
$7a+3y+4a+12-x$
$x-4+8y+6a+n$
$4a+7+3a+x-2y$ Ans.

8. Add $a-x+y+r-14-n$
$3n-y+2+6a-2+3x+r$
$62+7r-a-y-2n$
$3y+2a+1+6r+4n$
$x-7+x+y+4-n+r$. Ans.

TRANSPOSITION.

1. BY SUBTRACTION.

§45. It often happens that in the first member of the equation, some number has been added to the x's in order to make them equal to the last member. Thus, in the equation, $x+16=46$, we see that 16 has been added to x, to make it equal to 46.

§46. Now if x with 16 added, is equal to 46; then x alone must be 16 less than 46; that is, 46—16. So, that if we find, that $x+16=46$, we may know that $x=46-16$; or what is the same, $x=30$.

§47. But this may be proved another way. It is very plain that if we subtract a quantity from one member of an equation, and then subtract the same quantity from the other member of the equation; it will still be the fact that the two members are equal to one another. Thus, a half dollar= 50 cents. Subtract 2 cents from each member. Then a half dollar — 2 cents = 50 cents — 2 cents; for each of them is equal to 48 cents.

§48. Now, with the equation that we had above,

$$x+16=46.$$

Subtracting 16 from both members, $x+16-16=46-16$.

Now, in the first member of the equation, we have $+16$ -16, which is of no value at all, for $+16$ and -16 balance each other as has been seen in Ex. 1 under §28.

Therefore the equation is reduced to $\qquad x=46-16.$

Uniting terms in the last member, $\qquad x=30.$

EQUATIONS.—SECTION 2.

1. Suppose $x+8+3x=56$; what is the value of x?

Uniting terms, $\qquad 4x+8=56.$

Subtracting 8 from both, $4x+8-8=56-8$.
Which is the same as $4x=56-8$.
Uniting terms in the last member, $4x=48$.
Dividing by 4, $x=12$.

2. Suppose $2x+14-x-7=41+2-8$; to find x.
Uniting terms, $x+7=35$.
Subtracting 7 from both sides. $\left\{\begin{array}{l} x+7-7=35-7 \text{ or} \\ x=35-7. \end{array}\right.$
Uniting terms, $x=28$.

3. Given $x+5=6$, to find x.
Writing the equation, $x+5=6$.
Subtracting 5, $x=1$.

4. Given $5x+22-2x=31$, to find x. *Ans.* $x=3$.

5. Given $4x+20-6=34$, to find x. *Ans.* $x=5$.

6. Given $3x+12+7x=102$, to find x. *Ans.* $x=9$.

7. Given $10x-6x+14=62$, to find x. *Ans.* $x=12$.

8. If $7x-14+5x+20=246$, then $x=20$.

9. If $8x+17-5x+3=100+10$, then $x=30$.

10. If $7x-14+3x+35=450-29$, then $x=40$.

PROBLEMS.

1. What number is that, which, with 5 added to it, will be equal to 40?

Stating the question, x=the number
$x+5$,= after adding.
Forming the equation, $x+5=40$.
Subtracting 5, from both, $x=35$.

2. A man being asked how many shillings he had answered, add 15 to their number, and then subtract 1, and the remainder will be 64. How many shillings had he?

Stating the question, x=numb. of shillings.
$x+15$=after adding.
$x+15-1$=after subtracting.

Forming the equation, $x+15-1=64$.
Uniting terms, $x+14=64$.
Subtracting 14 from both, $x=50$.

3. What number is that, which with 9 added to it, will equal 23? *Ans.* 14.

4. Divide 17 dollars between two persons, so that one may have 4 dollars more than the other.

Stating the question, $x=$the less share.
$x+4=$the greater.
$x+x+4=$both shares.
Forming the equation, $x+x+4=17$.
Uniting terms, $2x+4=17$.
Subtracting 4 from both sides, $2x=13$.
Dividing by 2, $x=6\frac{1}{2}$.

5. The sum of the ages of a certain man and his wife is 55 years; and his age exceeds her's by 7 years. What is the age of each? ☞ Let $x=$age of the wife. *Ans.* 24 the wife's. 31 the man's.

6. A is 5 years older than B, and B is 4 years older than C; and the sum of their age is 73 years. What is the age of each?

Stating the question, $x=$C's age.
$x+4=$B's age.
$x+4+5=$A's age.
$x+x+4+x+4+5=$sum of all of them.
Forming the equation, $x+x+4+x+4+5=73$.
Uniting terms, $3x+13=73$.
Subtracting 13 from both sides, $3x=60$.
Dividing by 3, $x=20$.
Ans. C 20 years, B 24, A 29.

7. Two persons were candidates for a certain office, where there were 329 voters. The successful candidate

gained his election by a majority of 53. How many voted for each? *Ans.* 191 for one, and 138 for the other.

8. A, B, and C, would divide $200 among themselves, so that B may have $6 more than A; and C $8 more than B. How much must each have? *Ans.* A must have $60, B $76, and C $74.

9. Divide $1000 between A, B, and C: so that A shall have $72 more than B, and C $100 more than A. *Ans.* Give B $252, A $324, and C $424.

10. At a certain election 1296 persons voted, and the successful candidate had a majority of 120. How many voted for each? *Ans.* 588 for one, and 708 for the other.

11. A father, who has three sons, leaves them $8000, specifying in his will that the eldest shall have $1000 more than the second, and that the second shall have $500 more than the youngest. What is the share of each? *Ans.* The eldest had $3500, the second $2500, the youngest $2000.

12. A cask which held 74 gallons was filled with a mixture of brandy, wine and water. In it, there were 15 gallons of wine more than of brandy, and as much water as both wine and brandy. What quantity was there of each? *Ans.* 11 gallons of brandy, 26 of wine, and 37 of water.

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2. TRANSPOSITION BY ADDITION.

§49. It is frequently found that some quantity has been *subtracted* from the x's; as in the equation $5x-44=76$.

§50. In such cases, it is very evident, that the quantity which has been subtracted from the x's, must be *added* to

each side. For if in the above equation, 44 has already been subtracted from the $5x$; we must add it again if we wish to find what $5x$ is equal to. But if we add 44 to one member of the equation, we must also add as much to the other member of the equation. So that it will become

$$5x-44+44=76+44.$$

Uniting the two 44's, $5x=76+44.$

Or, $5x=120.$

Dividing by 5, $x=24.$

EQUATIONS.—SECTION 3.

1. Given $x+14+3x-27=51$, to find x. *Ans.* $x=16$.
2. Given $3x-30-2x=46-7$, to find x. *Ans.* $x=69$.
3. Given $9x-41+6=88-6$, to find x. *Ans.* $x=13$.
4. Given $20+3x-46=35-4$, to find x. *Ans.* $x=19$.
5. Given $4x-39-2x=47$, to find x. *Ans.* $x=43$.
6. Given $7x+27-46=65$, to find x. *Ans.* $x=12$.
7. Given $14x-55-8x+14=85$, to find x. *Ans.* $x=21$.

PROBLEMS.

1. What number is that, from which 8 being subtracted, the remainder is 45?

Stating the question $x=$ the number.

$x-8=$ when 8 is subtracted.

Forming the equation $x-8=45$

Adding 8 to both sides $x-8+8=45+8.$

Or, $x=53.$

2. What number is that, from which 27 being subtracted, the remainder is 41? *Ans.* 68.

3. A person bought two geese for \$1.40; and gave 16 cents more for one than he did for the other. What did each cost him?

☞ In this and the following questions of this section, x must stand for the greatest quantity.

Stating the question,	$x=$ the dearest.
	$x-16=$ the cheapest.
	$x+x-16=$ cost of both.
Forming the equation,	$x+x-16=140$
Uniting terms,	$2x-16=140$
Adding 16 to both sides,	$2x=140+16$
Uniting terms,	$2x=156$
Dividing by 2,	$x=78$

Ans. 78 cents, and 62 cents.

4. Three men, who are engaged in trade, put in \$2600 as follows: A put in a certain sum; B, \$60 less than A; and C, as much as A and B, lacking \$100. What was each man's share?

Ans. A's \$705; B's \$645; C's \$1250.

5. A purse of \$8000 is to be divided among A, B, and C; so that B may receive \$276 less than A, and C \$1112 less than A and B together. What is each man's share?

Ans. A's \$2416; B's \$2140; C's \$3444.

6. A father has willed to his four sons \$25200 as follows. To D a certain sum; to C as much as to D, lacking \$550; to B as much as to C, together with \$1550; and to A twice as much as to B, lacking \$10000. How much does each of them receive?

Ans. A \$5100; B \$7550; C \$6000; D \$6550.

7. Divide the number 60 into three such parts, that the first may exceed the second by 8, and the third by 16.

Ans. 12; 20; and 28.

8. Three men having found a purse of \$160, quarreled about the distribution of it. After the quarrel, it was found

that A had got a certain sum, and that B had $30 more than A, but C $50 less than A. How much did each obtain?

Ans. A $60; B $90; C $10.

III. TRANSPOSITION OF THE UNKNOWN QUANTITY.

§51. We have found that when any term has the sign $+$ it may be removed from one member of the equation to the other, if we take care to change the sign to $-$; for this has been done every time we have subtracted a term from both sides.

Thus, in the equation [$x+5=20$;] if we subtract 5 from both sides, it is plain that the first member becomes x, and the last member becomes $20-5$; so that the equation would become

$$x=20-5.$$

§52. So also any term that has the sign $-$ may be removed from one member to the other, if we take care to change the sign to $+$. Because this is the same as adding that term to both sides.

Thus, in the equation $x-5=20$,

if we add 5 to both sides, the first member becomes x, and the last member becomes $20+5$. So that the equation becomes

$$x=20+5.$$

§53. When we remove a term from one member of an equation to the other member, we say that we *transpose* that term; and the operation of doing it, is called *transposition*.

§54. It was stated in §32, that an equation must be brought so that the unknown quantity will occupy one member of the equation, and the known quantities embrace the other member. And, as it frequently happens that the

unknown quantities are on both sides, we are obliged to resort to transposition in order to make one side free from them. And likewise, it is often necessary to transpose *known* quantities from the member which contains the unknown quantity.

§55. Any term may be transposed from one member of an equation to the other, care being taken *to change the sign when we change the side.*

EQUATIONS.—SECTION 4.

1. Reduce the equation $4x-14=3x+12$

Solution. Transposing $3x$, $4x-3x-14=12$

Transposing 14, $4x-3x=12+14$

Uniting terms, $x=26$.

§56. In transposing, it is generally best to write first the *unknown quantity* that is on the left; and then bring over those which are on the right, if there are any there. Then write those *known* quantities that are *already* in the right hand member, and then transpose after them what known quantities there are in the left.

2. Given $21-7x=40-11x$, to find x. Ans. $x=4\frac{3}{4}$.

3. Given $40-6z=136-14z$, to find z. Ans. $z=12$.

4. Given $y+12=3y-4$, to find y. Ans. $y=8$.

5. Given $5x-15=2x+6$, to find x. Ans. $x=7$.

6. Given $40-6x-16=120-14x$, to find x.

Ans. $x=12$.

7. Given $4-9y=14-11y$, to find y. Ans. $y=5$.

8. Given $x+18=3x-5$, to find x.

Solution. Transposing $3x$, $x-3x+18=-5$

Transposing 18, $x-3x=-5-18$

Uniting terms, $-2x=-23$

Dividing by 2, $-x=-11\frac{1}{2}$.

§57. It is of no consequence what sign accompanies the final result; as the *magnitude* of the quantity is not affected by the sign. If we remember that $+$ is understood and may be written with every positive quantity, it will be very evident that the equation $-x=-11\frac{1}{2}$ is just as good as the equation $+x=+11\frac{1}{2}$. In both cases, the quantity x is equal to the number $11\frac{1}{2}$.

§58. In the result of the last question, $11\frac{1}{2}$ may be transposed to the first member; and x may be transposed to the last member. Of course, this will change the signs; and the equation will become $11\frac{1}{2}=x$. And if $11\frac{1}{2}=x$, it is evident that $x=11\frac{1}{2}$. This coincides with what was shown in §57.

§59. From what has just been said, we see that all the terms of each member may be transposed, so that the sign of each term may be changed; and still the equation shall retain the same members as at first; and that it is also immaterial which member is written first. And hence, in any equation *the signs of all the terms may be changed without affecting the equality.*

§60. It is evident that all the terms of one member may be transposed to the other member. When this has been done, the member *from which* the terms have been transposed becomes, 0. Thus, the equation $x=3y-xy$, may be made $x+xy-3y=0$; where $-3y$ balances $x+xy$.

PROBLEMS.

1. A man has six sons, whose successive ages differ by 4 years; and the eldest is three times as old as the youngest. What are their ages?

Stating the question, $x=$ age of the youngest.

$x+4=$ " " next

$x+4+4=$ " " next

$x+4+4+4=$ " " next

$x+4+4+4+4=$ " " next

$x+4+4+4+4+4=$ " " eldest.

Forming the equation, $x+4+4+4+4+4=3x$

Uniting terms, $x+20=3x$

Transposing the $3x$, $x-3x+24=0$

Transposing 20, $x-3x=-20$

Uniting terms, $-2x=-20$

Dividing by 2, $-x=-10$

or $x=10$

2. A person bought two horses, and also a hundred dollar harness. The first horse, with the harness, was of equal value with the second horse. But the second horse with the harness cost twice as much as the first. What was the price of each horse?

Stating the question, $x=$ price of the first.

$x+100=$ price of the second.

$x+100+100=$ 2d horse harnessed.

Forming the equation, $x+100+100=2x$

Transposing from both members, $x-2x=-100-100$

Uniting terms, $-x=-200$

Or $x=200$ &c.

3. A privateer running at the rate of 10 miles an hour, discovers a ship 18 miles off sailing at the rate of 8 miles an hour. How many hours can the ship run before she will be overtaken by the privateer?

☞ The equation will be $10x=8x+18$. Ans. 9 hours.

4. A gentleman distributing money among some poor

people, found that he would lack 10 shillings if he undertook to give 5s to each. Therefore, he gave only 4s to each, and finds that he has 5s left. How many persons were there.

☞ It will be found that his money by the first supposition $=5x-10$; and by the last supposition, it $=4x+5$.

Ans. 15.

5. I once had \$84 in my possession; and I gave away so much of it, that I have now three times as much as I gave away. How much did I give away?

☞ If I gave away \$$x$, then \$$84-x$ will be what remains. Ans. \$21.

6. A certain sum of money was shared among five persons, A, B, C, D, and E. Now, B received \$10 less than A; C \$16 more than B; D \$5 less than C; E \$15 more than D. And it was found that the shares of the last two put together, were equal to the sum of the shares of the other three. How much did each man receive?

Ans. A \$21; B \$11; C \$27; D \$22; E \$37.

7. A person wishes to give 3 cents apiece to some beggars, but finds that he has not money enough by 8 cents. He gives them 2 cents apiece and has 3 cents left. How many beggars were there? Ans. 11.

8. A courier who had started from a certain place 10 hours ago, is pursued by another from the same place, and on the same road. The first goes 4 miles an hour, and the second 9. In how many hours will the second overtake the first?

☞ In the operation, it must be remembered how far the first had the start, before the equal time for both began.

Ans. 8 hours.

MULTIPLICATION OF COMPOUND QUANTITIES BY SIMPLE QUANTITIES.

§61. Suppose you purchase 8 melons at 7 cents apiece, and afterwards find that you must give 5 cents apiece more for them. In this case you pay 8 times 7 cents, and also 8 times 5 cents; that is, first, 56 cents, and afterwards 40 cents.

§62. Let us apply this principle to Algebra. You pay in all, 8 times $7+5$, which $=56+40$. Which shows that *in multiplying a compound quantity, you multiply each term by itself.*

We can easily see that this operation will give the right answer; for in the case of the melons, they cost 12 cents apiece, and therefore their whole cost was 8 times 12 cents which $=96$ cents. But the answer just obtained, $56+40=96$.

§63. But suppose that after you had paid 7 cents apiece, a deduction of 5 cents apiece was made. The whole cost would then be 8 times $7-5$, which $=56-40$. And this agrees with the truth; for you first paid 56 cents, and afterwards 40 cents were deducted.

§64. *This shows that $+$ multiplied by $+$, produces $+$; and $-$ multiplied by $+$, produces $-$.*

EXAMPLES.

1. Multiply $x+4$, by 3. Ans. $3x+12$.
2. Multiply $12+x$, by 5. Ans. $60+5x$.
3. Multiply $x-10$, by 8. Ans. $8x-80$.
4. Multiply $126-x$, by 4. Ans. $504-4x$.
5. Multiply $x+8$, by 6.

6. Multiply $40+x$, by 10.

7. Multiply $x-32$, by 9.

8. Multiply $52-x$, by 12.

9. Multiply $2x+14$, by 7.

10. Multiply $27+3x$, by 14.

11. Multiply $3x-62$, by 15.

12. Multiply $97-4x$, by 12.

13. Multiply $x+7-y$, by 7.

14. Multiply $3x+y-12$, by 8.

15. Multiply $2x-3y-6$, by 6.

16. Multiply $3x-12+y$, by 5.

§65. Franciscus Vieta, a Frenchman, introduced about the year 1600, the *vinculum* or a straight line drawn over the top of two or more quantities when it is wished to connect them together. Thus, $\overline{x+4}\times3$, signifies that both x and 4 are to be multiplied by 3.

§66. In 1629, Albert Girard, a Dutchman, introduced the *parenthesis* as a convenient substitute, in many cases, for the vinculum. Thus, $(x+4)\times3$, is the same as $\overline{x+4}\times3$; and is read, $x+4$, *both* $\times3$. If there are more than two terms under the vinculum, we say, after repeating those terms, *all*, &c. Thus, $(x+y)\times(a-b+c)$, is read $x+y$ *both into* $a-b+c$ *all*. See also §100.

EQUATIONS.—SECTION 5.

1. $\overline{x-9}\times11=121$, to find x.

Solution, $\overline{x-9}\times11=121$

Performing the multiplication, $11x-99=121$

Transposing and uniting, $11x=220$

Dividing by 11, $x=20$.

2. Given $(x+7)\times 6=54$, to find x. Ans. $x=2$.

3. Given $\overline{12+x}\times 5=100$, to find x. Ans. 8.

4. Given $\overline{x-9}\times 8=96$, to find x. Ans. 21.

5. Given $\overline{181-3x}\times 5=920$, to find x. Ans. 61.

6. Given $(8+x)\times 2+14=72$, to find x. Ans. 21.

7. Given $(15+x)\times 3-27=48$, to find x. Ans. 10.

8. $(112-2x)\times 3=(2x-7)\times 4$, to find x. Ans. 26.

9. $(3x+14)\times 4=(78-x)\times 5$, to find x. Ans. $19\frac{9}{17}$.

10. $\overline{2x+8}\times 5=(32+x)\times 3$, to find x. Ans. 8.

11. $(3x-14)\times 7=(17-x)\times 6$, to find x. Ans. $14\frac{1}{3}$.

12. $\overline{120-3x}\times 2=(4x-6)\times 9$, to find x. Ans. 7.

PROBLEMS.

1. Two persons, A and B, lay out equal sums of money in trade; A gains \$126, and B loses \$87; and now A's money is double of B's. What did each lay out?

Stating the question, x,= the sum for each.
$x+126=$ A's sum now.
$x-87=$ B's sum now.
$2x-174=$ the double of B's.

Forming the equation, $x+126=2x-174$
Transposing and uniting, $-x=-300$
Changing signs, $x=300$ the answer.

2. A person, at the time he was married, was 3 times as old as his wife; but after they had lived together 15 years, he was only twice as old. What were their ages on their wedding day?

Stating the question, $x=$ the wife's age.

$3x=$ the man's age.

$x+15=$ the wife's after 15 years.

$3x+15=$ the man's after 15 years.

$2x+30=$ twice the wife's age.

Forming the equation, $3x+15=2x+30$

Transposing and uniting, $x=15$ the wife's age.

$3x=45$ the man's age.

3. A man having some calves and some sheep, and being asked how many he had of each sort; answered that he had twenty more sheep than calves, and that three times the number of sheep was equal to seven times the number of calves. How many were there of each?

☞ If $x=$ number of calves, there $x+20=$ number of sheep. Ans. 15 calves, and 35 sheep.

4. Two persons, A and B, having received equal sums of money, A paid away \$25, and B paid away \$60; and then it appeared that A had just twice as much money as B. What was the sum that each received? Ans. \$95.

5. Divide the number 75 into two such parts, so that three times the greater may exceed 7 times the less by 15.

☞ If $x=$ the greater, thus $75-x=$ the less; and $3x$ will $=7$ times the less $+15$. Ans. 54 and 21.

6. The garrison of a certain town consists of 125 men, partly cavalry and partly infantry. The monthly pay of a horse soldier is \$20, and that of a foot soldier is \$15; and the whole garrison receives \$2050 a month. What is the number of cavalry, and what of infantry?

☞ If $x=$ number of cavalry, then $20x=$ their whole pay, &c. Ans. 35 cavalry, and 90 infantry.

7. A grocer sold his brandy for 25 cents a gallon more than he asked for his wine; and 37 gallons of his wine

came to as much as 32 gallons of his brandy. What was each per gallon?

Ans. $1.60 for wine; and $1.85 for brandy.

8. A wine merchant has two kinds of wine; the one costs 9 shillings a gallon, the other 5. He wishes to mix both wines together, so that he may have 50 gallons that may be sold without profit or loss for 8 shillings a gallon. How much must he take of each sort?

☞ The whole mixture will be worth 50 times 8 shillings.

Ans. $37\frac{1}{2}$ gallons of the best; and $12\frac{1}{2}$ of the poorer.

9. A gentleman is now 40 years old, and his son is 9 years old. In how many years, if they both live, will the father be only twice as old as his son?

☞ In x years he will be $40+x$, and his son $9+x$.

Ans. 22 years.

10. A man bought 20 oranges and 25 lemons for $1.95. For each of the oranges he gave 3 cents more than for a lemon. What did he give apiece for each?

Ans. 3 cents for lemons, 6 cents for oranges.

11. A man sold 45 barrels of flour for $279; some at $5 a barrel, and some at $8. How many barrels were there of each sort? Ans. 27 at $5; and 18 at $8.

12. Says John to William, I have three times as many marbles as you. Yes, says William; but if you will give me 20, I shall have 7 times as many as you. How many has each?

☞ Let $x=$ William's and $3x=$John's. Then after the change, $x+20=$ William's and $3x-20=$ John's.

Ans. John 24; William 8.

13. A person bought a chaise, horse, and harness, for $440. The horse cost him the price of the harness, and

$30 more; and the chaise cost twice the price of the horse. What did he give for each?

Ans. For the harness $50; horse $130; chaise $260.

14. Two men talking of their ages, the first says your age is 18 years more than mine, and twice your age is equal to three times mine. What is the age of each?

Ans. Youngest 36 years. Eldest 54 years.

15. A boy had 41 apples which he wished to divide among three companions as follows; to the second, twice as many as to the first, and 3 apples more; and to the third, three times as many as to the second, and 2 apples more. How many did he give to each?

Ans. To the first 3; second 9; third 29.

16. How many gallons of wine, at 9 shillings a gallon, must be mixed with 20 gallons at 13 shillings, so that the mixture may be worth 10 shillings a gallon?

Ans. 60 gallons.

17. Two persons, A and B, have each an annual income of $400. A spends, every year, $40 more than B; and, at the end of 4 years, they both together save a sum equal to the income of either. What do they spend annually?

Ans. A $370; B $330.

FRACTIONS.

§65. All the division which the pupil has as yet performed, has been the division either of numeral quantities, or of the numeral co-efficients. But in Algebra, it is frequently necessary to divide *literal* quantities. For example, after having made x to stand for an unknown quantity, we may wish to find the *half* of x, or the *third* of x, or the *fourth* of x, &c.

§66. In common arithmetic, if we wish to divide 1 by 2, we do it by writing 2 under the 1; thus, $\frac{1}{2}$. So if we wish to divide 2 by 3, we write 3 under the 2; thus, $\frac{2}{3}$. In the same manner, 2 divided by 5 is written $\frac{2}{5}$; $3 \div 4$ is written $\frac{3}{4}$; $6 \div 7$ is written $\frac{6}{7}$. The quantities that are obtained by dividing in *this manner*, are called *fractions.*

§67. In Algebra, we most generally make use of this method of dividing; especially when we divide *literal* quantities. Or, in other words, *we divide a literal quantity by writing the divisor under the dividend, with a straight line between them;* thus, x divided by 2, is written $\frac{x}{2}$; and is read, *x-half.* $x \div 3$, is written $\frac{x}{3}$; and is read, *x-third;* $x \div 4$, is written $\frac{x}{4}$, and is read *x-fourth;* $3x \div 4$, $\frac{3x}{4}$, and is read, *3x-fourth.*

§68. The two separate numbers that we employ in writing a fraction, are called *terms.* The upper term is called the *numerator*, and the lower term is called the *denominator.* Thus, in the fraction $\frac{x}{3}$, we call x the numerator, and 3 the denominator.

§69. If the one-third of x is $\frac{x}{3}$, it is evident that $\frac{2}{3}$ of x, is two times as much; that is $\frac{2x}{3}$. If $\frac{1}{5}$ of x is $\frac{x}{5}$, then $\frac{3}{5}$ of

5

x is $\frac{3x}{5}$. Whence the rule *to multiply a whole number by a fraction, is, to multiply the whole number by the numerator, and divide by the denominator;* as $\frac{4}{9}$ of x is $\frac{4x}{9}$; $\frac{3}{7}$ of y is $\frac{3y}{7}$; $\frac{2}{3}$ of a is $\frac{2a}{3}$.

Examples. The pupil may multiply a, x, and y, each of them by $\frac{2}{3}$; and then by $\frac{3}{4}$; and then by $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{5}{7}$, $\frac{3}{7}$, successively.

§70. As we can multiply a number of *parts* as well as a number of *wholes*, and as the denominator is nothing more than the *name* of the parts; it is plain, that *to multiply a fraction, we multiply the numerator, and retain the denominator without alteration.* Thus, 2 times $\frac{3}{5}$ is $\frac{6}{5}$; 3 times $\frac{5}{6}$ is $\frac{15}{6}$; 2 times $\frac{x}{3}$ is $\frac{2x}{3}$; 4 times $\frac{2a}{3}$ is $\frac{8a}{3}$; &c.

Examples. Multiply each of the following fractions by 2, then by 3, and then by 4. $\frac{2}{3}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{7}$, $\frac{6}{7}$, $\frac{x}{2}$, $\frac{x}{3}$, $\frac{x}{4}$, $\frac{x}{5}$, $\frac{2x}{3}$, $\frac{3x}{4}$, $\frac{2x}{5}$, $\frac{3x}{5}$, $\frac{4x}{7}$.

§71. As we know that 2 halves = a whole, we readily conclude that 4 halves = 2 wholes; and that 6 halves = 3 wholes; &c. Likewise, because 3 thirds = a whole, 6 thirds must equal 2 wholes; and 9 thirds must equal 3 wholes. In the same manner 8 fourths = 2 wholes; 20 fifths = 4 wholes; 18 thirds = 6 wholes; &c. Such fractions are called *improper fractions.*

§72. Hence, in order to find how many *whole ones* there are in any number of halves, we have only to see how many times *two halves* are contained in that number. Thus, in 10 halves there are as many whole ones as there are 2 halves contained in 10 halves; which is 5. In the same manner, in 12 thirds there are as many whole ones as there are 3 thirds contained in 12 thirds; which is 4.

§73. Thus we have the rule, *to change an improper*

fraction to a whole number, divide the numerator by the denominator.

When the answer consists of an integer and a fraction, it is called a *mixed number.*

Examples, 1. How many whole ones in $\frac{8}{3}$.

Ans. $8 \div 3 = 2\frac{2}{3}$.

2. How many whole ones in $\frac{7}{2}$? $\frac{12}{5}$? $\frac{14}{3}$? $\frac{20}{4}$? $\frac{25}{5}$?

3. How many whole ones in $\frac{20}{2}$? $\frac{26}{3}$? $\frac{29}{4}$? $\frac{36}{5}$? $\frac{40}{6}$?

4. How many whole x's in $\frac{6x}{2}$? $\frac{12x}{3}$? $\frac{20x}{4}$? $\frac{27x}{3}$?

5. How many whole x's in $\frac{10x}{2}$? $\frac{30x}{3}$? $\frac{48x}{4}$? $\frac{50x}{5}$?

6. How many whole x's in 3 times $\frac{2x}{6}$?

7. How many whole x's in 4 times $\frac{x}{2}$?

8. How many whole x's in 5 times $\frac{3x}{5}$?

§74. If we have the quantity $\frac{x}{5}$, we know that, as it takes 5 fifths to make a whole one, it will take 5 times this quantity to make a whole x. Therefore, if we multiply $\frac{x}{5}$ by 5, we shall obtain $\frac{5x}{5}$, or exactly x. If we multiply $\frac{x}{3}$ by 3, we shall obtain $\frac{3x}{3}$, or, which is the same, x. If we multiply $\frac{x}{4}$ by 4, we shall obtain $\frac{4x}{4}$, or x.

§75. As 4 times $\frac{1x}{4}$ is equal to x; then 4 times $\frac{2x}{4}$ must be equal to twice as much, or $2x$; and 4 times $\frac{3x}{4}$ must be three times as much, or $3x$. As 3 times $\frac{x}{3}$ is equal to x; so 3 times $\frac{2x}{3}$ must be twice as much, or $2x$. So 5 times $\frac{3x}{5}$ must be 3 times as much as 5 times $\frac{x}{5}$; and therefore is $3x$.

§76. *Any fraction when multiplied by the number which is the same as the denominator, will produce a quantity*

which is the same as the numerator. Thus,

$$\frac{5x}{4}\times4=5x; \qquad \frac{7x}{3}\times3=7x.$$

We shall be able to make use of this principle in the solution of many equations, if we operate in accordance with the following *axiom* or self-evident truth.

§77. *If equals be multiplied by the same, their products will be equal.* Thus, if $x=10$, then $2x=20$; $4x=40$; &c.

EQUIVALENT FRACTIONS.

§78. It is evident [§71,] that each of the following fractions, $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, $\frac{5}{5}$, $\frac{6}{6}$, $\frac{7}{7}$, $\frac{8}{8}$, $\frac{9}{9}$, &c., is equal to 1. Therefore, they must be equal to one another. Also, each of the following fractions, $\frac{2}{1}$, $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, $\frac{12}{9}$, &c., is equal to 2; and consequently they are all equal to one another. In the same manner, we may make many fractions that will equal 3; and so of any other number.

§79. Let us take from the first set of the above fractions, $\frac{2}{2}$ and $\frac{4}{4}$ which are equal to one another. We see that both the numerator and denominator in the last fraction are twice as much as in the first. We see the same fact in the equal fractions $\frac{3}{3}$ and $\frac{6}{6}$; and also in the equal fractions $\frac{4}{4}$ and $\frac{8}{8}$. We find the same, by taking from the second set, the equal fractions $\frac{2}{1}$ and $\frac{4}{2}$; and also $\frac{4}{2}$ and $\frac{8}{4}$; and also $\frac{6}{3}$ and $\frac{12}{6}$.

§80. Again in the equal fractions, $\frac{2}{2}$ and $\frac{6}{6}$, we find each term in the last fraction three times as great as the correspondent term in the first fraction. The same may be observed in the fractions $\frac{3}{3}$ and $\frac{9}{9}$; and also in $\frac{2}{1}$ and $\frac{6}{3}$; and also in $\frac{4}{2}$ and $\frac{8}{4}$.

§81. By pursuing this investigation, we shall find that *whenever we multiply both the numerator and the denominator by the same number*, no matter what that number may be, *the fraction made by that multiplication, will be equal to the first fraction.* Hence, there is an equality between the following fractions, $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$, &c.; and also between the following, $\frac{1}{3}$, $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, $\frac{5}{15}$, $\frac{6}{18}$, &c.

§82. The principle just explained, leads to another which is of much importance. Suppose we multiply by 8, both terms of the fraction $\frac{2}{3}$; and obtain $\frac{16}{24}$. It is plain that both terms of the fraction $\frac{16}{24}$ can be divided by 8, to bring the fraction back to $\frac{2}{3}$. So also, if both terms in $\frac{3}{4}$ be multiplied by 6, the fraction will be $\frac{18}{24}$, which means just as much as $\frac{3}{4}$; and, of course, if both terms in $\frac{18}{24}$ be divided by 6, the fraction will be brought back to $\frac{3}{4}$, which is equal to $\frac{18}{24}$. So, in general, *if we divide both the numerator and the denominator of a fraction by the same number, we have a new fraction which will be equal to the first.* Thus, $\frac{8}{10}$ may be changed to $\frac{4}{5}$; $\frac{12}{18}$ to $\frac{2}{3}$; $\frac{15}{21}$ to $\frac{5}{7}$.

§83. It is evident, that of several fractions of equal value, that which has the least denominator is the most easily understood. Thus, $\frac{5}{6}$ of an apple is much better known at first sight, than $\frac{35}{42}$ of an apple. And *when a fraction is brought to as small a denominator as it can be changed to, we say it is reduced to its lowest terms.*

§84. In order to reduce a fraction to its lowest terms, *divide both the numerator and the denominator by any number that will divide each without a remainder.* Thus, in the fraction $\frac{75}{105}$, both terms may be divided by 5, by which we obtain $\frac{15}{21}$; and both terms of this last fraction may be divided by 3, by which we obtain $\frac{5}{7}$.

EQUATIONS.—SECTION 6.

Problems.

1. In an orchard, $\frac{1}{4}$ of the trees bear apples, $\frac{1}{5}$ of them bear pears, $\frac{2}{11}$ bear plums, and 81 bear cherries. How many trees are there in the orchard; and how many of each sort?

Stating the question, $x=$ number of trees.

$\frac{x}{4}=$ apple trees.

$\frac{x}{5}=$ pear trees.

$\frac{2x}{11}=$ plum trees.

$81=$ cherry trees.

All these trees together $=$ the whole orchard.

Forming the equation, $\frac{x}{4}+\frac{x}{5}+\frac{2x}{11}+81=x$.

Now, we know that if we multiply $\frac{x}{4}$ by 4 we obtain x alone; that is, we *destroy the fraction*, and make it a whole number. And we know, that if we multiply the first member by 4; and also the last member by 4, we shall not destroy the *equation*. See §77. We will therefore multiply both members by 4, for the purpose of destroying the *fraction* in the first term. It will then become

$$x+\frac{4x}{5}+\frac{8x}{11}+324=4x.$$

Next, we will multiply both members by 5, to destroy the fraction in the second term. This will make

$$5x+4x+\frac{40x}{11}+1620=20x.$$

Then we will multiply by 11, to destroy the remaining fraction, which will make

$$55x+44x+40x+17820=220x.$$

Transposing and uniting, $-81x=-17820$.

Changing signs, $81x=17820$.

Dividing by 81, $x=220$. the Ans.

2. In a certain school, $\frac{1}{5}$ of the boys learn mathematics, $\frac{3}{4}$ of them study Latin and Greek, and 6 study English grammar. What is the whole number of scholars?

☞ After the question is stated, the equation will become $\frac{x}{5}+\frac{3x}{4}+6=x$

Multiplying by 5, $x+\frac{15x}{4}+30=5x$

Multiplying by 4, $4x+15x+120=20x$

Transposing and uniting, $-x=-120$. Ans.

3. A gentleman has an estate, $\frac{1}{6}$ of which is woodland, $\frac{2}{3}$ of it pasture, and 105 acres embrace the pleasure grounds, gardens, and orchards. How many acres does it contain?

Ans. 630 acres.

4. After paying away $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I find 22 dollars yet in my purse. How much had I at first?

Ans. \$40.

5. A man bought a lot of ground, for which he agreed to pay as follows: $\frac{1}{4}$ of the money on taking possession, $\frac{1}{3}$ of it in 6 months, and \$250 at the end of the year. How much did he pay in all? Ans. \$600.

6. A post is one-fourth of its length in the mud, one-third in the water, and 10 feet above the water. What is its whole length? Ans. 24 feet.

7. In a Christmas pudding, $\frac{1}{4}$ is flour, $\frac{1}{5}$ milk, $\frac{1}{6}$ eggs, $\frac{1}{3}$ suet and fruit, and $\frac{3}{4}$ of a pound of spices and other ingredients. What is the weight of the pudding?

☞ The equation will be $\frac{x}{4}+\frac{x}{5}+\frac{x}{6}+\frac{x}{3}+\frac{3}{4}=x$.

Ans. 15 pounds.

8. A lady being asked what her age was; replied, if you add $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ of my age together, the sum will be 18. How old was she?

☞ After the question has been stated, the equation will be, $\frac{x}{3}+\frac{x}{4}+\frac{x}{6}=18$. Ans. 24 years.

9. What sum of money is that, whose $\frac{1}{3}$ part, $\frac{1}{4}$ part, and $\frac{1}{5}$ part, added together, amount to 94 dollars? Ans. \$120.

10. A person found upon beginning the study of his profession, that he had passed $\frac{1}{7}$ of his life before he commenced his education, $\frac{1}{3}$ of it under a private teacher, the same time at a public school, and four years at the university. What was his age? Ans. 21 years.

11. How much money have I in my pocket, when the fourth and fifth part of it together, amount to \$2.25? Ans. \$5.

12. The 3d part of my income, said a person, I expend in board and lodging, the 8th part of it in clothes and washing, the 10th part of it in incidental expenses, and yet I save \$318 a year. What was his yearly income? Ans. \$720.

13. A gentleman bequeaths, in his will, the half of his property to his wife, one-sixth part to each of his two sons, the twelfth part to his sister, and the remaining \$600 to his servant. What was the amount of his property? Ans. \$7200.

14. Of a piece of metal, $\frac{1}{3}$ plus 24 ounces is brass, and $\frac{3}{4}$ minus 42 ounces is copper. What is the weight of the piece?

☞ The equation, when formed, will be

$$\frac{x}{3}+24+\frac{3x}{4}-42=x.$$

Ans. 216 oz.

15. A farmer mixes a quantity of grain, so that 20 bushels less than $\frac{1}{2}$ of it is barley, and 36 bushels more

than $\frac{1}{3}$ of it is oats. How many bushels are there in the whole; and how many of each sort?

☞ In stating the question; $\frac{x}{2}-20=$ barley, and $\frac{x}{3}+36=$ oats.

Ans. 96 bushels in all; 28 of barley, and 68 of oats.

16. A teacher being asked how many scholars he had, replied, If I had as many more, half as many more, and quarter as many more, I should have 88. How many had he?

☞ In stating the question, he has x; and as many more is another x; &c. Ans. 32.

17. In a mixture of copper, tin, and lead; 16 pounds less than $\frac{1}{2}$ was copper, 12 pounds less than $\frac{1}{3}$ was tin, and 4 pounds more than $\frac{1}{4}$ was lead. What was the weight of the whole mixture; and also of each kind?

Ans. 288lb.; and also 128lb., 84lb., and 76lb.

18. What is that number whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part, by 12?

☞ The statement is the same as, $\frac{1}{3}$ of it equals $\frac{1}{4}$ of it $+12$. Ans. 144.

19. What number is that whose $\frac{1}{3}$ part exceeds its $\frac{1}{5}$ part by 72? Ans. 540.

20. A certain sum of money is to be divided amongst three persons, A, B, and C, as follows. A is to receive \$3000 less than half of it, B \$1000 less than the third of it, and C \$800 more than the fourth of it. What is the sum to be divided; and what does each receive?

Ans. \$38400; and also, \$16200, \$11800, \$10400.

21. A man driving his geese to market, was met by another, who said, Good morrow, master, with your hundred geese. He replied, I have not a hundred; but if I had as

many more, and half as many more, and two geese and a half, I should have a hundred. How many had he?

Ans. 39 geese.

22. A shepherd, being asked how many sheep he had, replied, If I had as many more, half as many more, and 7 sheep and a half, I should have just 500. How many sheep had he? Ans. 197 sheep.

23. A legacy of $1200 was left between A and B, in such a manner, that $\frac{1}{8}$ of A's share was equal to $\frac{1}{7}$ of B's. What sum did each receive? Ans. A $640; B $560.

24. If the half, third, and fourth parts of my number, be added together, the sum will be one more than my number. Now, what is my number? Ans. 12.

25. A says to B, your age is twice and $\frac{3}{5}$ of my age; and the sum of our ages is 54 years. What is the age of each? Ans. A's 15 years; B's 39 years.

26. A young gentleman having received a fortune, spent $\frac{1}{3}$ of it the first year, $\frac{1}{4}$ of it the second, and $\frac{1}{5}$ of it the third, when he had $2600 left. What was his whole fortune?

☞ In stating the question, what was spent, = the whole minus $2600. Ans. $12000.

27. A father leaves four sons who share his property in the following manner. The first takes half, minus $3000; the second takes a third, minus $1000; the third takes exactly a fourth; and the fourth son takes a fifth and $600. What was the whole fortune, and what did each son receive?

Ans. The whole fortune was $12000, and each son received $3000.

DIVISION OF COMPOUND QUANTITIES BY SIMPLE QUANTITIES.

§86. We have found, §67, that the algebraical method of dividing, is to write the divisor under the dividend, with a straight line between them. It is plain that compound quantities may be divided in this way, as well as simple quantities. Thus, $14+x$ is divided by 3, as follows:

$$\frac{14+x.}{3}$$

§87. With the same reason, we find the fraction of a compound number, by multiplying it by the numerator, and writing the denominator under the product. Thus,

$$\tfrac{2}{3} \text{ of } x-5 \text{ is } \frac{2x-10.}{3}$$

EXAMPLES.

1. What is $\frac{3}{4}$ of $8+x$? Ans. $\dfrac{24+3x.}{4}$

2. What is $\frac{2}{5}$ of $x-27$? Ans. $\dfrac{2x-54.}{5}$

3 What is $\frac{2}{7}$ of $3x-14$? Ans. $\dfrac{6x-28.}{7}$

4. What is $\frac{3}{8}$ of $9+5x$? Ans. $\dfrac{27+15x.}{8}$

5. What is $\frac{4}{7}$ of $7x-19$? Ans. $\dfrac{28x-76.}{7}$

6. What is $\frac{5}{9}$ of $9x-27$? Ans. $\dfrac{45x-135.}{9}$

§88. This may be changed into whole numbers by §73. Thus, $\dfrac{45x-135}{9}=5x-15.$

EQUATIONS.—SECTION 7.

Problems.

1. A young gentleman being asked his age, said it is such that if you add 8 years to it, and then divide by 3, the quotient would be 9. How old was he?

Stating the question, $x=$ his age.

$x+8=$ with 8 added.

Forming the equation, $\frac{x+8}{3}=9$

Multiplying by 3, $x+8=27$

Transposing and uniting, $x=19$ the Ans.

2. A man being asked what he gave for his horse, replied, that if he had given \$12 more, $\frac{3}{4}$ of the sum would be 84. What was the price of the horse?

Stating the question, $x=$ the price.

$x+12=$ when increased.

$\frac{3x+36}{4}=\frac{3}{4}$ of the sum.

Forming the equation $\frac{3x+36}{4}=84$

Multiplying by 4, $3x+36=336$

Transposing and uniting, $3x=300$

Dividing by 3, $x=100$ the Ans.

3. What sum of money is that, from which \$5 being subtracted, two-thirds of the remainder shall be \$40?

Ans. \$65.

4. It is required to divide a line that is 15 inches long into two such parts, that one of them may be $\frac{3}{4}$ of the other.

☞ In stating the question, the parts are x, and $15-x$.

Ans. $8\frac{4}{7}$; and $6\frac{3}{7}$.

5. It is required to find a number, such that if 15 be subtracted from it, $\frac{4}{5}$ of the remainder shall be 100?

Ans. 140.

6. Divide the number 46 into two parts, so that when one is divided by 7, and the other by 3, the quotients together may amount to 10. Ans. 28 and 18.

7. A person being asked the time of day, answered that the time past from noon was equal to $\frac{4}{5}$ of the time to midnight. What was the hour?

☞ In stating the question, the parts are x, and $12-x$.

Ans. 20 minutes after 5.

8. Two men talking of their horses, A says to B, my horse is worth \$25 more than yours; and $\frac{3}{5}$ of the value of my horse is equal to $\frac{3}{4}$ of the value of yours. What is the value of each?

☞ The values will be x, and $x+25$.

Ans. A's \$125; B's 100.

9. Two persons have equal sums of money. One having spent \$39, and the other \$93, the last has but half as much as the first. How much had each? Ans. \$147.

10. A man being asked the value of his horse and chaise, answered that the chaise was worth \$50 more than the horse; and that one half the value of the horse was equal to one third the value of the chaise. What was the value of each? Ans. Horse \$100; Chaise \$150.

11. What number is that, to which if I add 13, and from $\frac{1}{13}$ of the sum subtract 13, the remainder shall be 13?

Ans. 325.

12. A man being asked the value of his horse and saddle, answered that his horse was worth \$114 more than his sad-

dle, and that $\frac{2}{3}$ of the value of his horse was 7 times the value of his saddle. What was the value of each?

Ans. Saddle \$12; Horse \$126.

13. A person rented a house on a lease of 21 years, and agreed to do the repairs when $\frac{2}{3}$ of that part of the lease which had elapsed, should equal $\frac{8}{9}$ of the part to come. How long will he hold possession before he repairs?

Ans. 12 years.

14. What number is that, to which if I add 20, and from $\frac{2}{3}$ of this sum subtract 12, the remainder shall be 10? Ans. 13.

15. A person has a lease for 99 years; and being asked how much of it was already expired, he answered that $\frac{2}{3}$ of the time past was equal to $\frac{4}{5}$ of the time to come. What time had already past? Ans. 54 years.

16. Divide \$183 between two men, so that $\frac{4}{7}$ of what the first receives, shall be equal to $\frac{3}{10}$ of what the second receives. What will be the share of each?

Ans. \$63 and \$120.

17. Bought sheep for \$300, calves for \$100, and pigs for \$25, and then laid out \$15, which was $\frac{5}{6}$ of the rest of my money, in getting them home. How much had I at first? Ans. \$443.

18. A gentleman paid four labourers \$136. To the first he paid three times as much as to the second wanting \$4; to the third one half as much as to the first, and \$6 more; and to the fourth four times as much as to the third, and \$5 more. How much did he pay to each?

Ans. To the first \$26; second \$10; third \$19; fourth \$81.

DIVISION OF FRACTIONS, AND FRACTIONS OF FRACTIONS.

§90. It was shown in §70, that in multiplying a fraction, we multiply the numerator only, and retain the denominator. On the same principle, *a fraction is divided by dividing its numerator, and retaining its denominator.*

$$\text{Thus, } \frac{6}{7} \div 3 = \frac{2}{7}. \quad \frac{8x}{9} \div 4 = \frac{2x}{9}, \text{ \&c.}$$

§91. But, supposing we wish to divide $\frac{2}{7}$ by 3. In this case, we cannot divide the numerator 2 by 3 without a remainder; and therefore we must look for some other principle to assist us. We shall find it in §81, where it was shown that a fraction may be changed to one with different terms, without altering the value.

§92. It is evident then, that we have only to change the fraction which is to be divided, to some equivalent fraction, whose numerator can be divided by the divisor without a remainder. Thus, $\frac{2}{7}$ can be changed $\frac{6}{21}$, $\frac{12}{42}$, $\frac{18}{63}$, &c.; each of which can be divided by 3, giving for the quotient either $\frac{2}{21}$, or $\frac{4}{42}$, or $\frac{6}{63}$, &c.

§93. The *most convenient* equivalent fraction will be obtained by multiplying both terms of the fraction by the number which is to be the divisor. Because it is certain that after the numerator has been multiplied by a number, the product can in return be *divided* by that number.

$$\text{Thus, } \frac{4}{9} \div 5 = \frac{20}{45} \div 5; \text{ and } \frac{20}{45} \div 5, = \frac{4}{45}.$$

§94. But, by examining the example just given, we find the numerator of the answer to be the same as the numerator of the first fraction; for the first numerator has been multiplied and then divided again by the same number.

The *denominator only* is changed; and that has been done by multiplying the first denominator by the number that was to be the divisor.

§95. Hence the *rule* for dividing a fraction. *Divide its numerator when it can be done without a remainder. But if there should be a remainder, multiply the denominator by the divisor for a new denominator; and leave the numerator as it is.*

Thus, $\frac{2}{9} \div 5 = \frac{2}{45}$; $\frac{7}{11} \div 6 = \frac{7}{66}$.

EXAMPLES.

1. Divide $\frac{3x}{4}$ by 5. Ans. $\frac{3x}{20}$.

2. Divide $\frac{2x}{3}$ by 3. Ans. $\frac{2x}{9}$.

3. Divide $\frac{4x}{7}$ by 2. Ans. $\frac{2x}{7}$.

4. Divide $\frac{5x}{3}$ by 5. Ans. $\frac{x}{3}$.

5. Divide $\frac{12x}{5}$ by 3. Ans. $\frac{4x}{5}$.

6. Divide $\frac{7+x}{3}$ by 2. Ans. $\frac{7+x}{6}$.

7. Divide $\frac{x-6}{5}$ by 4. Ans. $\frac{x-6}{20}$.

8. What is $\frac{1}{7}$ of $\frac{3x}{4}$? Ans $\frac{3x}{28}$.

9. What is $\frac{1}{5}$ of $\frac{4x}{7}$? Ans. $\frac{4x}{35}$.

10. What is $\frac{1}{3}$ of $\frac{41-2x}{4}$? Ans. $\frac{41-2x}{12}$.

11. What is $\frac{1}{4}$ of $\frac{3x-47+y}{16}$? Ans. $\frac{3x-47+y}{64}$.

12. What is $\frac{1}{5}$ of $\frac{7x+4}{x-1}$? Ans. $\frac{7x+4}{5x-5}$.

13. What is $\frac{1}{7}$ of $\frac{21-3x}{2x+9}$? Ans. $\frac{21-3x}{14x-63}$.

14. What is $\frac{1}{6}$ of $\frac{48+7y}{3x-2y}$? Ans. $\frac{48+7y}{18x-12y}$.

☞ In this example, although we can divide the first term in the numerator, we do not do it because we cannot divide the last term.

15. What is $\frac{1}{9}$ of $\frac{27-6y}{31+x-y}$? Ans. $\frac{27-6y}{279+9x-9y}$.

16. What is $\frac{1}{8}$ of $\frac{32x+16}{x-6+y}$? Ans. $\frac{4x+2}{x-6+y}$.

§96. We are now enabled to find a fraction of a fraction by the rule in §69; which is to multiply by the numerator, and divide by the denominator. *To multiply by the numerator is to multiply the numerators together. And to divide by the denominator*, we have just shown, *is to multiply the denominators together.*

EXAMPLES.

1. What is $\frac{2}{3}$ of $\frac{5x}{6}$? Ans. $\frac{10x}{18}=\frac{5x}{9}$.

2. What is $\frac{3}{4}$ of $\frac{7x}{8}$? Ans. $\frac{21x}{32}$.

3. What is $\frac{4}{5}$ of $\frac{3x}{7}$? Ans. $\frac{12x}{35}$.

4. What is $\frac{2}{7}$ of $\frac{5x-1}{3}$? Ans. $\frac{10x-2}{21}$.

5. What is $\frac{3}{5}$ of $\frac{71+2x}{4}$? Ans. $\frac{273+6x}{20}$.

6. What is $\frac{4}{9}$ of $\frac{6x}{23+y}$? Ans. $\frac{24x}{207+9y}$

7. What is $\frac{2}{11}$ of $\frac{9y}{x+61}$? Ans. $\frac{18y}{11x+671}$.

8. What is $\frac{9}{10}$ of $\frac{4x-20}{41+6x}$? Ans. $\frac{36x-180}{410+60x}=\frac{18x-90}{205+30x}$.

9. What is $\frac{4}{7}$ of $\frac{3y+7x}{2x-11y}$? Ans. $\frac{12y+28x}{14x-77y}$.

10. What is $\frac{3}{10}$ of $\frac{6x+7y-10}{4+3y-2x}$? Ans. $\frac{18x+21y-30}{40+30y-20x}$.

11. What is $\frac{4}{9}$ of $\frac{x-y-6}{6+x-y}$? Ans. $\frac{4x-4y-24}{54+9x-9y}$.

12. What is $\frac{2}{5}$ of $\frac{3x+5y-8}{27-4x}$? Ans. $\frac{6x+10y-16}{135-20x}$.

EQUATIONS.—SECTION 8.

1. A farmer wishes to mix 116 bushels of provender, consisting of rye, barley, and oats, so that it may contain $\frac{5}{7}$ as much barley as oats, and $\frac{1}{2}$ as much rye as barley. How much of each must there be in the mixture?

Stating the question, $x=$oats; and $\frac{5x}{7}=$ barley.

Then, $\frac{1}{2}$ of $\frac{5x}{7}=\frac{5x}{14}=$ rye.

Forming the equation, $x+\frac{5x}{7}+\frac{5x}{14}=116$.
Multiplying by 14, $14x+10x+5x=1624$
Uniting terms, $29x=1624$
Dividing by 29, $x=56$ the Ans.

2. I paid away a fourth of my money, and then a fifth of the remainder, which was \$72. How much money had I at first?

Stating the question, $x=$ what I have.

$\frac{x}{4}=$ paid first.

$x-\frac{x}{4}=$ the remainder.

$\frac{x}{5}-\frac{x}{20}=$ paid afterwards.

Forming the equation, $\frac{x}{5}-\frac{x}{20}=72$

Multiplying by 20, $4x-x=1440$

Uniting terms, $3x=1440$

Dividing by 3, $x=480$ the Ans.

3. After paying away $\frac{1}{4}$ of my money, and then $\frac{1}{5}$ of the remainder, I had \$72 left. How much had I at first?

In stating the question, the remainder after the first payment was $\frac{3x}{4}$; and $\frac{1}{5}$ of that is $\frac{3x}{20}$.

Forming the equation, $x-\frac{x}{4}-\frac{3x}{20}=72$

Multiplying by 20, $20x-5x-3x=1440$

Uniting terms, and dividing, $x=120$

4. A clerk spends $\frac{2}{3}$ of his salary for his board, and $\frac{2}{3}$ of the remainder in clothes, and yet saves \$150 a year. What is his yearly salary? Ans. \$1350.

5. Of a detachment of soldiers, $\frac{2}{3}$ are on actual duty, $\frac{1}{8}$ sick, $\frac{1}{5}$ of the remainder absent on leave, and there are 380 officers. What is the number of men in the detachment? Ans. 1000 men.

6. A young man, who had just received a fortune, spent $\frac{3}{8}$ of it the first year, and $\frac{4}{5}$ of the remainder the next year; when he had \$1420 left. What was his fortune? Ans. \$11360.

7. If from $\frac{1}{3}$ of my height in inches, 12 be subtracted, $\frac{1}{5}$ of the remainder will be 2. What is my height? Ans. 5 ft. 6 in.

8. A bowl of punch was mixed as follows: $\frac{1}{6}$ was rum,

$\frac{1}{4}$ brandy, $\frac{1}{12}$ acid and sugar, and 3 pints more than half of all these was water. How much did the bowl contain?

Ans. 6 quarts.

9. A, B, and C, own together a field of 36 acres. B has $\frac{1}{3}$ more than A, and C has $\frac{1}{4}$ more than B. What is each man's share? Ans. A, 9 acres; B, 12; C, 15.

10. A gentleman leaves \$315 to be divided among four servants in the following manner. B is to receive as much as A, and $\frac{1}{2}$ as much more; C is to receive as much as A and B, and $\frac{1}{3}$ as much more; D is to receive as much as the other three, and $\frac{1}{4}$ as much more. What is the share of each? Ans. A \$24; B \$36: C \$80; D \$175.

SUBTRACTION OF COMPOUND QUANTITIES.

§97. Suppose we wish to subtract the expression $x+6$ from y. It is evident that we may first subtract x; which will give us, $y-x$. But we wish to subtract not only x, but 6 also. Well, after we have subtracted x we will subtract 6 also; and then the answer will be, $y-x-6$. Therefore, whenever we wish to subtract a compound quantity whose terms are all positive, we write them after the other quantity with $+$ changed to $-$.

§98. Again, suppose we wish to subtract the expression $x-6$ from y; in which the number to be subtracted has $-$ instead of $+$. As before, we will first subtract x, by which we obtain $y-x$. But the quantity to be subtracted was 6 less than x; and we have therefore subtracted 6 too much. We will therefore add 6 to our last answer for the true remainder, which will give us $y-x+6$. Here, we have changed the positive x to $-$; and the -6, to $+6$.

§99. Hence the propriety of the following *rule* for subtracting compound quantities. *Change all the signs of the expression which is to be subtracted, the sign + to —, and the sign — to +; and then write the terms after the other quantity.* It is to be recollected that in the quantity *from which* we subtract, the signs are not altered.

§100. Although the subtraction is performed the moment the quantities are written according to the above rule, yet it is always expedient to *unite the terms* if necessary after the operation.

EXAMPLES.

1. Subtract $7x+6+y$ from $6y-17$.
   Ans. $6y-17-7x-6-y$; which $=5y-23-7x$.
2. Subtract $4y+3x-10$ from $74-x$.
   Ans. $74-x-4y-3x+10$; which $=84-4x-4y$.
3. Subtract $6-x-3y$ from $7x+6y$.
   Ans. $7x+6y-6+x+3y$; which equals $8x+9y-6$.
4. From $4x-3y+27$, subtract $6y-12+x$.
   Ans. $3x-9y+39$.
5. From $6+x-y$, subtract $13-9y-x$.
   Ans. $2x+8y-7$.
6. From $8x-2+3y$, subtract $3y+4-3x$.
   Ans. $11x-6$.
7. From $5+4x$, subtract $2-5x+4y-z$.
   Ans. $9x-4y+3+z$.
8. From $6x-8y$, subtract $-x-y+50$.
   Ans. $7x-7y-50$.
9. From $14-\frac{2x}{3}$, subtract $3x-12$.
   Ans. $26-3x-\frac{2x}{3}$.

10. From $\frac{8+7x}{3}$, subtract $5+\frac{3x}{5}$.

Ans. $\frac{8+7x}{3}-5-\frac{3x}{5}$.

11. From $7-\frac{2y}{3}$, subtract $\frac{10y-4}{6}$.

Ans. $7-\frac{2y}{3}-\frac{10y-4}{6}$.

§101. It has been shown, §65 and §66, that any compound quantity may be considered and operated upon as a simple quantity, by merely drawing a vinculum above it, or enclosing it in a parenthesis. Whenever that compound quantity is a fraction, the *line between the numerator and denominator serves as a vinculum.* Thus, in the eleventh example, above, $\frac{10y-4}{6}$ is subtracted as a simple quantity; and therefore the sign in it is not changed.

12. From $\frac{4x-6}{4}$, subtract $\frac{3x+7}{5}$.

Ans. $\frac{4x-6}{4}-\frac{3x+7}{5}$.

13. From $\frac{3x+8}{2}$, subtract $\frac{51-x}{3}$.

Ans. $\frac{3x+8}{2}-\frac{51-x}{3}$.

14. From $\frac{7-2x}{3}$, subtract $\frac{21x-4}{10}-x$.

Ans. $7-\frac{2x}{3}-\frac{21x-4}{10}+x$.

EQUATIONS.—SECTION 9.

1. There are two numbers, whose sum is 140; and if 4 times the less be subtracted from 3 times the greater, the remainder is 70. What are the numbers?

Stating the question, $x=$ the greater.
$140-x=$ the less.
Forming the equations, $3x-560+4x=70$.
Transposing and uniting, $x=90$.
Ans. Greater 90; Less 50.

2. A person, after spending \$100 more than a third of his yearly income, found he had \$150 more than half of it remaining. What was his income? Ans. \$1500.

3. Divide the number 48 into two such parts, that the excess of one of them above 20, shall be three times as much as the other wants of 20. Ans. 32 and 16.

4. Two men, A and B, commenced trade. A had twice as much money as B; he has since gained \$50, and B has lost \$90; and now the difference between A's and B's money, is equal to three times what B has. How much had each when they commenced trade?
Ans. A \$205; B \$410.

5. A gentleman bought a watch and chain for \$160. If $\frac{3}{4}$ of the price of the watch be subtracted from six times the price of the chain, the remainder will be the same as if $\frac{5}{12}$ of the price of the chain were subtracted from twice the price of the watch. What was the price of each?
Ans. Watch \$112; Chain \$48.

6. Divide the number 204 into two such parts, that if $\frac{2}{5}$ of the less were subtracted from the greater, the remainder will be equal to $\frac{3}{7}$ of the greater subtracted from four times the less. Ans. Greater 154; Less, 50.

7. Two travellers, A and B, found a purse of money. A first takes out \$2 and $\frac{1}{6}$ of what remains; and then B takes out \$3 and $\frac{1}{6}$ of what remains; and it is found that each have the same sum. How much money was in the purse?
Ans. \$20.

8. A man bought a horse and chaise for \$341. If $\frac{3}{8}$ of the price of the horse be subtracted from twice the price of the chaise, the remainder will be the same as if $\frac{5}{7}$ of the price of the chaise be subtracted from three times the price of the horse. What was the price of each?

☞ If the chaise be x, and the horse be $341-x$; then the first remainder will be $2x-\frac{1023-3x}{8}$. But when the fraction is destroyed, the vinculum is taken away and therefore the last sign must be changed from $-$ to $+$.

Ans. Chaise \$189; Horse \$152.

9. A person in play lost a fourth of his money, and then won back 3*s.*; after which he lost a third of what he now had, and then won back 2*s.*; lastly he lost a seventh of what he then had, and then found he had but 12*s.* remaining. What had he at first? Ans. 20*s.*

10. A lady spent $\frac{4}{7}$ of her money at the linen drapers, $\frac{7}{9}$ of the remainder at the mercers, $\frac{1}{2}$ of what she had left at the milliners, paid 3*s.* for a coach, and carried home $\frac{1}{30}$ of the sum she had at first. How much had she at first?

Ans. 210 shillings.

UNITING FRACTIONS OF DIFFERENT DENOMINATORS.

§102. By looking at the answers to the last three examples in §100, and also the three in §101, it will appear that we ought to have some rule for uniting their terms. We can easily find one by applying the principle explained in §81. For we have only to change each of the fractions to such equivalent fractions as will have one with another the same denominator.

Thus, the answer to the twelfth example under §100, is $\frac{4x-6}{4}-\frac{3x+7}{5}$. Now each of these two fractions may be changed to 20ths, by multiplying the first by 5, and the last by 4. They will then become $\frac{20x-30}{20}-\frac{12x+28}{20}$ which $=\frac{(20x-30)-(12x+28)}{20}=\frac{20x-30-12x-28}{20}$ $=\frac{8x-58}{20}$.

§103. Thus we have the *rule* for uniting fractions of different denominators. *Multiply all the denominators together for a new denominator; and each numerator by all the denominators except its own, for new numerators; remembering that if a compound numerator follows —, all the signs in it must be changed the moment one denominator is used for the whole quantity.*

EXAMPLES.

1. Unite the terms in the answer to the 13th sum in §101.

Operation.

$$\frac{3x+8}{2}-\frac{51-x}{3}=\frac{9x+24}{6}-\frac{102-2x}{6}=$$
$$\frac{9x+24-102+2x}{6}=\frac{11x-78}{6}.$$

Note.—In this operation the first minus has reference to the whole quantity $\left(\frac{51-x}{3}\right)$; the second minus to the whole quantity $\left(\frac{102-2x}{6}\right)$. In this last quantity, 102 has no sign before *itself*, and is therefore positive. Now, when the line between the numerator and denominator is carried *through the whole quantity*, the vinculum of $102-2x$ is

destroyed; and then is the time for changing the signs for subtracting.

2. Unite the terms in the quantity $7-\frac{2y}{3}-\frac{10y-4}{6}$.

Operation. Here we have 7 which $=\frac{7}{1}$, so that the quantity is the same as $\frac{7}{1}-\frac{2y}{3}-\frac{10y-4}{6}$. Both terms of the fraction $\frac{7}{1}$ multiplied by 3 times $6=\frac{126}{18}$. $\left.\frac{2y}{3}\right\}\times 6=\frac{12y}{18}$. $\left.\frac{10y-4}{6}\right\}\times 3=\frac{30y-12}{18}$

$\therefore$* $\frac{7}{1}-\frac{2y}{3}-\frac{10y-4}{6}=\frac{126}{18}-\frac{12y}{18}-\frac{30y-12}{18}$, which $=\frac{126-12y-30y+12}{18}$; which $=\frac{138-42y}{18}=\frac{23-7y}{3}$.

3. Unite the terms in the quantity, $\frac{8+7x}{3}-5-\frac{3x}{5}$.

Ans. $\frac{40+35x-75-9x}{15}=\frac{26x-35}{15}$.

4. Unite the terms in the quantity, $26-3x-\frac{2x}{3}$.

Ans. $\frac{78-9x-2x}{3}=\frac{78-7x}{3}=\frac{78}{3}-\frac{7x}{3}=26-\frac{7x}{3}$.

5. Unite the terms in the quantity, $7-\frac{2x}{3}-\frac{21x-4}{10}$ $+x$. Ans. $\frac{222-53x}{30}$.

6. Unite the terms in the quantity, $12-\frac{6+2x}{4}-\frac{4x-3}{5}$ $-\frac{12x}{8}$. Ans. $\frac{111-28x}{10}$.

* $\therefore$ Signifies *therefore.*

7. Unite the terms in the quantity, $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}$.

Ans. $\frac{13x}{12}=x+\frac{x}{12}$.

8. Unite the terms in the quantity, $4+\frac{x}{3}-3+\frac{2x}{9}$.

Ans. $1+\frac{x}{9}$.

9. Unite the terms in the quantity, $4z+\frac{3z}{7}+z-\frac{4z}{5}$.

Ans. $5z-\frac{13z}{35}$.

10. Unite the terms in the quantity, $y-\frac{3y-4}{5}+2y-\frac{6+2y}{3}$.

Ans. $\frac{26y-3}{15}=y+\frac{11y-8}{15}$.

§104. Whenever there are integers to be united with fractions, *they* may be changed to fractions, by putting the number 1 under them for the denominator. See example 2.

Thus, $6=\frac{6}{1}$; $8=\frac{8}{1}$.

RATIO AND PROPORTION.

§105. When an unknown quantity is not, either by itself, or in some connexion with others, known to be *equal* to some known quantity or set of quantities; we may sometimes find that there is a *comparison* between it and some known quantity, which is the same as the comparison between two *known* quantities.

Thus, suppose I buy 27 yards of cloth for $72, and wish to sell for $16 so much of it as cost me $16. In this case the number of yards to be sold is not equal to any other quantity that is mentioned. But we suppose that it must compare with the number of yards bought, in the same manner that $16 compares with $72. By knowing this comparison we can find the number of yards; because, as $16 is $\frac{2}{9}$ of $72, so the number of yards to be sold must be $\frac{2}{9}$ of the number of yards bought. It is 6 yards.

§106. It will be seen that the comparison in this example, consists in observing how many times one of the numbers is contained in the other. 16 is contained in 72, two ninths of a time. When a comparison of this kind is made, the result that is obtained is called their RATIO. Thus, in comparing the numbers 3 and 4, we find that 4 is contained in 3, three-fourths of a time; and therefore we say *the ratio of* 3 *to* 4 *is* $\frac{3}{4}$.

§107. The pupil must remember that the ratio of one number to another, always signifies how the *first* number compares with the *last.* Thus, the ratio of 8 to 5, is $\frac{8}{5}$; that is, 8 is $\frac{8}{5}$ of 5. Hence the ratio is expressed by making the *first* term to be a *numerator*, and the *last* to be the *denominator*.

§108. In the example just furnished relative to the cloth; the ratio of the money paid, to the money obtained for a part of the cloth; (that is, the ratio of \$72 to \$16,) is $\frac{72}{16}=\frac{9}{2}$. And so also the ratio of the cloth bought, to the cloth sold; (that is, the ratio of 27 yards to 6 yards,) is $\frac{27}{6}$ which equals $\frac{9}{2}$. Here we see, that although the ratios are differently expressed, they are, notwithstanding, equal to one another.

§109. When the ratio of two quantities is equal to the ratio of other two quantities, there is said to be a PROPORTION between them; that is, *an equality of ratios is called a proportion.*

§110. Our chief business with ratios at present, is to learn when they form a proportion; that is, when they are equal to one another. Now, as they may be expressed in the form of a fraction, it is evident, that when they are brought to a common denominator, if they are proportional their numerators will be the same, and if they are not proportional their numerators will not be the same.

For example, is 11 to 21=33 to 63? We pursue our inquiry as follows: 11 to 21 is the same as $\frac{11}{21}$, and 33 to 63 is the same as $\frac{33}{63}$. We bring the fractions to a common denominator by §103.

$$\frac{11}{21} \text{ and } \frac{23}{63} = \frac{693}{1323} \text{ and } \frac{693}{1323}.$$

We find they are equal, and the four terms 11 to 21=33 to 63 are proportional.

§111. Although ratios are sometimes expressed fractionally, they are generally expressed as follows: 11 : 21 and 33 : 63; that is 11 divided by 21, 33 divided by 63. The pupil will see that we employ the same sign that expresses

division, with the exception of the — between the two dots. The sign : is read *is to*, and the foregoing examples are read 11 *is to* 21 and 33 *is to* 63.

§112. When four quantities are proportional, they are written thus, 11 : 21 : : 33 : 63. The sign : : is read *as*; and the whole expression is read, 11 is to 21 as 33 is to 63.

§113. In a proportion, the first and the last terms are called *extremes*, and the two middle terms are called *means*. In the above proportion, 11 and 63 are the extremes, and 21 and 33 are the means.

§114. In order to derive any important use from a proportion, we wish the pupil to recollect the method employed to find whether four quantities are proportional. We multiplied, (see §110,) the first numerator by the last denominator, to find one new numerator. These were the two *extremes*. We also multiplied the last numerator by the first denominator, to find the other new numerator. These were the two *means*. And hence we learn, that *if four quantities are proportional, the product of the two extremes is equal to the product of the means.*

§115. Therefore, whenever in our operations we find a proportion, we can easily reduce it to an equation by multiplying the extremes together for one member; and multiplying the means together for the other member. Thus, $2:7::8:x$, becomes in an equation $2x=56$; whence $x=28$.

EQUATIONS.—SECTION 10.

1. If you divide $75 between two men in the proportion of 3 to 2, what will each man receive?

Stating the question, $x=$ the share of one.
$75-x=$ the share of the other.
Making the proportion, $x:75-x::3:2$
Reducing to an equation, $2x=225-3x$
Transposing and uniting, $5x=225$
Dividing, $x=45$

Ans. \$45; and \$30.

2. Divide 150 into two parts, so that the parts may be to each other as 7 to 8. Ans. 70; and 80.

3. Divide \$1235 between A and B, so that A's share may be to B's as 3 to 2.
Ans. A's share \$741; B's \$494.

4. Two persons buy a ship for \$8640. Now the sum paid by A is to that paid by B, as 9 to 7. What sum did each contribute? Ans. A paid \$4860; B \$3780.

5. A prize of \$2000 was divided between two persons, whose shares were in proportion as 7 to 9. What was the share of each? Ans. \$875; and \$1125.

6. A gentleman is now 30 years old, and his youngest brother 20. In how many years will their ages be as 5 to 4?

☞ After stating the question the proportion will be $30+x:\ 20+x::5:4$. Ans. 20 years.

7. What number is that, which, when added to 24, and also to 36, will produce sums that will be to each other as 7 to 9? Ans. 18.

8. Two men commenced trade together. The first put in \$40 more than the second; and the stock of the first was to that of the second as 5 to 4. What was the stock of each? Ans. \$200; and \$160.

9. A gentleman hired a servant for \$100 a year, together with a suit of clothes which he was to have immedi-

ately. At the end of 8 months, the servant went away, and received \$60 and kept the suit of clothes. What was the value of the suit of clothes? Ans. \$20.

10. A ship and a boat are descending a river at the same time; and when the ship is opposite a certain fort, the boat is 13 miles ahead. The ship is sailing at the rate of 5 miles while the boat is going 3. At what distance below the fort, will they be together? Ans. $32\frac{1}{2}$ miles.

§116. It is very often the case that a problem is easily solved by using simply the *ratio*, instead of a proportion.

EQUATIONS.—SECTION 11.

Operation by Ratio.

1. Divide 40 apples between two boys in the proportion of 3 to 2.

Stating the question, $x=$ the share of one.

Now, as the ratio of the first to the second is $\frac{3}{2}$; then the ratio of the second to the first is $\frac{2}{3}$. Therefore,

$$\frac{2x}{3}= \text{the share of the second.}$$

Forming the equation, $x+\frac{2x}{3}=40$

Multiplying by 3, $3x+2x=120$

Consequently, $x=24$.

Ans. 24; and 16.

2. Three men trading in company, gain \$780. As often as A put in \$2, B put in \$3, and C put in \$5. What part of the gain must each of them receive?

Stating the question, $x=$ A's share.

$$\frac{3x}{2}= \text{B's share.}$$

$$\frac{5x}{2}= \text{C's share.}$$

Forming the equation, $x+\frac{3x}{2}+\frac{5x}{2}=780.$

Ans. A \$156; B \$234; C \$390.

3. Two butchers bought a calf for 40 shillings, of which the part paid by A, was to the part paid by B, as 3 to 5. What sum did each pay? Ans. A paid 15*s.*; B 25*s.*

4. Divide 560 into two such parts, that one part may be to the other as 5 to 2. Ans. 400; and 160.

5. A field of 864 acres is to be divided among three farmers, A, B, and C; so that A's part shall be to B's as 5 to 11, and C may receive as much as A and B together. How much must each receive?

Ans. A 135; B 297; C 432 acres.

6. Three men trading in company, put in money in the following proportion: the first 3 dollars as often as the second 7, and the third 5. They gain \$960. What is each man's share of the gain? Ans. \$192; \$448; \$320.

7. Find two numbers in the proportion of 2 to 1, so that if 4 be added to each, the two sums will be in proportion of 3 to 2. Ans. 8 and 4.

8. Two numbers are to each other as 2 to 3; but if 50 be subtracted from each, one will be one half of the other. What are the numbers? Ans. 100 and 150.

9. A sum of money is to be divided between two persons, A and B; so that as often as A takes \$9, B takes \$4. Now it happens that A receives \$15 more than B. What is the share of each? Ans. A \$27; B \$12.

10. There are two numbers in proportion of 3 to 4; but if 24 be added to each of them, the two sums will be in the proportion of 4 to 5. What are the numbers?

Ans. 72 and 96.

11. A man's age when he was married was to that of his wife as 3 to 2; and when they had lived together 4 years, his age was to hers as 7 to 5. What were their ages when they were married?

Ans. His age 24; hers 16 years.

12. A certain man, found when he married, that his age was to that of his wife as 7 to 5. If they had been married 8 years sooner, his age would have been to hers as 3 to 2. What were their ages at the time of their marriage?

Ans. His age 56 years; hers, 40.

13. A man's age, when he was married, was to that of his wife as 6 to 5; and after they had been married 8 years, her age was to his as 7 to 8. What were their ages when they were married? Ans. Man 24; Wife 20 years.

14. A bankrupt leaves \$8400 to be divided among four creditors, A, B, C, and D, in proportion to their claims. Now, A's claim is to B's as 2 to 3; B's claim to C's as 4 to 5; and C's claim to D's as 6 to 7. How much must each creditor receive?

Ans. A \$1280; B \$1920; C \$2400; D \$2800.

15. A sum of money was divided between two persons, A and B, so that the share of A was to that of B as 5 to 3. Now, A's share exceeded $\frac{5}{9}$ of the whole sum by \$50. What was the share of each? Ans. \$450; and \$270.

EQUATIONS WITH TWO UNKNOWN QUANTITIES.

§117. It frequently happens, that *several* unknown quantities are introduced into a problem. But when this is the case, if the conditions will give rise to as many equations, independent of each other, as there are unknown quantities, there is no difficulty in finding the value of each quantity.

§118. An equation is said to be independent of another when it cannot be changed into that other. Thus, $7x-y=47$, is independent of the equation $10y+4x=50$; because one of them cannot be so altered as to make the other. But, $7x-y=47$, is not independent of the equation $21x-3y=141$; because the last is made by multiplying the first by 3.

§119. At present we will attend only to equations that include *two* unknown quantities, *each* represented by a *different letter from the other*.

§120. In equations that contain two unknown quantities, our first object must be to find the value of one of them; and in order to do this, the preliminary step is to derive from the equations that are given, another equation which shall have but one unknown quantity. This operation is called *exterminating the other unknown quantities.*

§121. There are three different methods of forming *one* equation with *one unknown quantity* from two equations containing two unknown quantities. With each of these, the learner should become familiar; as it is sometimes convenient to use one of them, and sometimes another.

First Method.

§122. It is necessary here to recollect what was stated in §42, that when equals are added to equals, their sums will be equal; and also when equals are subtracted from equals, their remainders are equal. Thus, suppose we have the equation, $x+14=36$; and suppose also that we know that $y=8$; then if we will add y to the first member, and 8 to the last, the members will still be equal to one another, as follows: $x+14+y=36+8$. And also if we subtract y from the first member, and 8 from the last, the members will still be equal to one another; thus $x+14-y=36-8$.

§123. This principle can be easily applied for the extermination of unknown quantities. For, if in both of two equations, one of the unknown quantities has the *same coefficient*, but after *different signs*, it is evident that if we add both equations together, viz. the first member to the first member, and the last member to the last member; and then unite terms, a new equation will be formed in which that unknown quantity will disappear.

EXAMPLES.

1. Given the two equations, $\left\{\begin{matrix}3x+2y=26\\5x-2y=38\end{matrix}\right\}$ to find the values of x and y.

Add together the two right hand members, and also the two left; and we have the equation

$$3x+2y+5x-2y=26+38$$

Uniting terms, $8x=64$

Dividing, $x=8$

Now, if $x=8$, then $3x=24$; and the first equation will become $24+2y=26$, from which we may find the value of y.

2. Given the equations, $\left\{\begin{matrix}7x+4y=58\\9x-4y=38\end{matrix}\right\}$ to find the values of x and y. Ans. $x=6$; $y=4$.

3. Given the equations, $\left\{\begin{matrix}5x+6y=58\\2x+6y=34\end{matrix}\right\}$ to find the values of x and y.

☞ In this example, it is plain that the y's will not be destroyed by adding them together. But we have before seen, §59, that if all the signs are changed, the equation will not be affected. Let us then change the signs of the second equation. The two equations will then become $\left\{\begin{matrix}5x+6y=58\\-2x-6y=-34\end{matrix}\right\}$ which, when added together, become $5x+6y-2x-6y=58-34$, or $3x=24$. $\therefore x=8$; and the first equation becomes $40+6y=58$. Whence, $y=3$.

§124. In the last example, if we take the equations before the alteration of the second, thus, $\left\{\begin{matrix}5x+6y=58\\2x+6y=34\end{matrix}\right\}$ and subtract the second from the first, the result will be the same as it was by changing the signs. As follows: $5x+6y-2x-6y=58-34$.

Whence we learn that, if in both equations one of the unknown quantities has the *same co-efficient* and also the *same sign*, and we subtract one equation from the other, (viz. the first member from the first member, and the second member from the second member,) and unite the terms; we shall form a new equation in which that unknown quantity will disappear.

EXAMPLES.

4. Given the equations, $\left\{\begin{matrix}6x+7y=79\\6x+3y=51\end{matrix}\right\}$ to find the values of x and y? Ans. $x=5$; $y=7$.

5. Given $\begin{cases} 5x-6y=64 \\ 2x-6y=58 \end{cases}$ to find x and y.

Ans. $x=2$; $y=9$.

6. Given $\begin{cases} x-3y=73 \\ x-6y=106 \end{cases}$ to find x and y.

Ans. $x=40$; $y=11$.

7. Given $\begin{cases} 12x+8y\ =92 \\ 12x-21y=63 \end{cases}$ to find x and y.

Ans. $x=7$; $y=1$.

8. Given $\begin{cases} 2x+2y=18 \\ 3x-2y=\ 7 \end{cases}$ to find x and y.

Ans. $x=5$; $y=4$.

9. Given $\begin{cases} 4x+3y=\ \ 22 \\ -4x+2y=-12 \end{cases}$ to find x and y.

Ans. $x=4$; $y=2$.

10. Given $\begin{cases} 3x+5y=40 \\ x+2y=14 \end{cases}$ to find x and y.

☞ In this example, neither of the unknown quantities has the same co-efficient in both equations. But both members of the last equation can be multiplied by 3, without destroying the equality, §77; and then the x's will be alike in both equations. Thus, $\begin{cases} 3x+5y=40 \\ 3x+6y=42 \end{cases}$

Ans. $x=10$; $y=2$.

11. Given $\begin{cases} 6x+5y=128 \\ 3x+4y=\ 88 \end{cases}$ to find x and y.

☞ The second may be multiplied by 2.

Ans. $x=8$; $y=16$.

12. Given $\begin{cases} x+2y=17 \\ 3x-\ y=\ 2 \end{cases}$ to find x and y.

☞ Make the y's alike. Ans. $x=3$; $y=7$.

13. Given $\begin{cases} 2x-\ y=6 \\ 4x+3y=22 \end{cases}$ to find x and y.

☞ Multiply the first by 2 and the x's will be alike.

Ans. $x=4$; $y=2$.

14. Given $\left\{\begin{array}{r} 2+3z=38 \\ 6x+5z=82 \end{array}\right\}$ to find x and z.

Ans. $x=7$; $z=8$.

15. Given $\left\{\begin{array}{l} 4x+6y=46 \\ 5x-2y=10 \end{array}\right\}$ to find x and y.

Ans. $x=4$; $y=5$.

16. Given $\left\{\begin{array}{l} 2x+3y=31 \\ 4x-3y=17 \end{array}\right\}$ to find x and y.

Ans. $x=8$; $y=5$.

17. Given $\left\{\begin{array}{r} 4y+z\ =102 \\ y+4z=48 \end{array}\right\}$ to find y and z.

Ans. $y=24$; $z=6$.

18. Given $\left\{\begin{array}{l} 2x+3y\ \ =7 \\ 8x-10y=6 \end{array}\right\}$ to find x and y.

Ans. $x=2$; $y=1$.

19. Given $\left\{\begin{array}{l} 5y+3x=93 \\ 3y+4x=80 \end{array}\right\}$ to find y and x.

Multiply the first by 4, $\quad 20y+12x=372$

Multiply the second by 3, $\quad 9y+12x=240$

Subtracting the 2d from the 1st, $\quad 11y \quad =132$.

Ans. $y=12$; $x=11$.

20. Given $\left\{\begin{array}{l} 4y-5z=2 \\ 5y-4z=7 \end{array}\right\}$ to find y and z.

Multiply the first by 5, $\quad 20y-25z=10$

Multiply the second by 4, $\quad 20y-16z=28$

Subtracting the 1st from the 2d, $\quad 9z=18$.

Ans. $y=3$; $z=2$.

§125. From the foregoing, we derive the following :

RULE I. TO EXTERMINATE AN UNKNOWN QUANTITY.

Determine which of the unknown quantities you will exterminate ; and then, if it is necessary, multiply or divide one or both of the equations so as to make the term which contains that unknown quantity to be the same in both.

Then if the identical terms have LIKE *signs in both*

equations, SUBTRACT *one equation from the other; but if they have* UNLIKE *signs,* ADD *one equation to the other.* And the result will be an equation containing only *one* unknown quantity.

EQUATIONS.—SECTION 12.

1. What two numbers are those whose sum is 20 and difference 12?

Stating the question, $x=$ greater number.
$y=$ the less.

Then forming the equations, $x+y=20$
$\underline{x-y=12}$

Adding the equations, $2x=32 \therefore x=16$

Substituting 16 for x in the first, $16+y=20$

Transposing and uniting, $y=4$.

Ans. 16 and 4.

2. A market woman sells to one person, 3 quinces and 4 melons for 25 cents; and to another, 4 quinces and 2 melons, at the same rate, for 20 cents. How much are the quinces and melons apiece?

Forming the equations, $3x+4y=25$
$4x+2y=20$

Multiplying the second by 2, $\underline{8x+4y=40}$

Subtracting 1st from 2d, $5x=15$.

Ans. Quinces 3 cents; Melons 4.

In any of our solutions after this, we shall number the lines, so that any reference to them will be easily understood.

3. A man bought 3 bushels of wheat and 5 bushels of rye for 38 shillings; and at another time 6 bushels of wheat and 3 bushels of rye for 48 shillings. What was the price for a bushel of each?

Let $x=$ price of wheat, and $y=$ price of rye.

1. By the first condition, $3x+5y=38$
2. By the second, $6x+3y=48$
3. Multiplying the 1st by 2, $\underline{6x+10y=76}$
4. Subtracting the 2d from the 3d, $7y=28 \therefore y=4$
5. Substituting 4 for y in the 1st, $3x+20=38$
6. Transposing and dividing, $x=6.$

Ans. Wheat for 6*s.*; Rye for 4*s.*

4. Two purses together contain \$400. If you take \$40 out of the first and put them into the second, then there is the same in each. How many dollars does each contain?

Let $x=$ the number in the first.
$y=$ the number in the second.

1. By the first condition, $x+y=400$
2. By the second, $x-40=y+40$
3. Transposing the 2d, $\underline{x-y=80}$
4. Adding the 1st to the 3d, $2x=480 \therefore x=240.$

Ans. The first \$240; the second \$160.

5. A gentleman being asked the age of his two sons, replied that if to the sum of their ages 25 be added, this sum will be double the age of the eldest; but if 8 be taken from the difference of their ages, the remainder will be the age of the youngest. What is the age of each?

Let $x=$ the age of the eldest. $y=$ the age of the youngest.

1. By the first condition, $x+y+25=2x$
2. By the second, $x-y-8=y$
3. Transposing and uniting 1st, $-x+y=-25$
4. Transposing and uniting 2d, $\underline{x-2y=8}$
5. Adding 3d and 4th, $-y=-17$
6. Substituting 17 for y in the 3d, $-x+17=-25$
7. Transposing $-x=-42$

Ans. Eldest 42; Youngest 17.

6. A gentleman paid for 6 pair of boots and 4 pair of shoes \$44; and afterwards for 3 pair of boots and 7 pair of shoes \$32. What was the price of each per pair?

Ans. Boots \$6; Shoes \$2.

7. A man spends 30 cents for apples and pears, buying his apples at the rate of 4 for a cent, and his pears at the rate of 5 for a cent. He afterwards let his friend have half of his apples and one-third of his pears for 13 cents, at the same rate. How many did he buy of each sort?

Let $x =$ number of apples.

$y =$ number of pears.

$\frac{1}{4}$ cent $=$ price of 1 apple.

$\frac{1}{5}$ cent $=$ price of 1 pear.

$\frac{x}{4}$ cents $=$ price of all the apples.

$\frac{y}{5}$ cents $=$ price of all the pears.

1. By the first condition, $\frac{x}{4}+\frac{y}{5}=30$

2. By the second, $\frac{x}{8}+\frac{y}{15}=13$

3. Dividing the 1st by 3, $\frac{x}{12}+\frac{y}{15}=10$

4. Subtracting 3d from 2d, $\frac{x}{8}-\frac{x}{12}=3$

5. Multiplying by 24, $3x-2x=72 \therefore x=72.$

Ans. 72 apples; 60 pears.

8. One day a gentleman employs 4 men and 8 boys to labour for him, and pays them 40*s.*; the next day he hires at the same rate, 7 men and 6 boys, for 50*s.* What are the daily wages of each? Ans. Man's, 4*s.*; boy's, 2*s* 6*d.*

9. It is required to find two numbers with the following properties. $\frac{1}{2}$ of the first with $\frac{1}{3}$ of the second shall make 16: and $\frac{1}{4}$ of the first with $\frac{1}{5}$ of the second shall make 9.

Ans. 12 and 30.

10. Says A to B, give me 5*s*. of your money, and I shall have twice as much as you will have left. Says B to A, give me 5*s*. of your money, and I shall have three times as much as you will have left. What had each?

Ans. A 11*s*; B 13*s*.

11. Two men agree to buy a house for \$1200. Says A to B, give me $\frac{2}{3}$ of your money, and I shall be able to pay for it all; No, says B, give me $\frac{3}{4}$ of yours, and then I can pay for it. How much money had each?

Ans. A \$800; B \$600.

12. Find two numbers with the following properties. The products of the first by 2, and the second by 5, when added are equal 35. Also, the products of the first by 7, and the second by 4, when added are equal to 68.

Ans. 8 and 3.

13. A paid B 20 guineas, and then B had twice as much money as A had left; but if B had paid A 20 guineas, A would have had three times as much as B had left. What sum did each possess at first?

Ans. A 52 guineas; B 44.

14. A person has a saddle worth £50, and two horses. When he saddles the poorest horse, the horse and saddle are worth twice as much as the best horse; but when he saddles the best, he with the saddle is worth three times the poorest. What is the value of each horse?

Ans. Best £40; Poorest £30.

15. A merchant sold a yard of broadcloth and 3 yards of velvet for \$25; and, at another time, 4 yards of broadcloth

and 5 yards of velvet for \$65. What was the price of each per yard? Ans. Broadcloth \$10; Velvet \$5.

16. A person has 500 coins consisting of eagles and dimes; and their value amounts to \$1931. How many has he of each coin?

☞ The solution must be in *cents*.

Ans. 190 eagles; 310 dimes.

17. In the year 1299, three fat oxen and 6 sheep together cost 79 shillings; and the price of an ox exceeded the price of 12 sheep by 10 shillings. What was the value of each? Ans. An ox 24 shillings; a sheep 1*s.* 2*d.*

18. Two persons talking of their ages, A says to B, 8 years ago I was three times as old as you were; and 4 years hence, I shall be only twice as old as you. What are their present ages? Ans. A 44; B 20 years.

19. A farmer sold to one man 30 bushels of wheat and 40 of barley for 270 shillings; and to another, 50 bushels of wheat and 30 of barley for 340 shillings. What was the price per bushel of each? Ans. Wheat 5*s.*; Barley 3*s.*

20. A man and his wife and child dine together at an inn. The landlord charged 15 cents for the child, and for the woman he charged as much as for the child, and $\frac{1}{3}$ as much as for the man; but for the man he charged as much as for the woman and child together. What did he charge for each?

Ans. 45 cents for the man; and 30 cents for the woman.

21. A gentleman has two horses, and also a chaise worth \$250. If the first horse be harnessed, he and the chaise will be worth twice as much as the second horse; but if the second be harnessed, he and the chaise will be worth three times as much as the first horse. What is the value of each horse? Ans. First \$150; second \$200.

22. A is in debt \$1200, and B owes \$2500; but neither has enough to pay his debts. A says to B, lend me the $\frac{1}{8}$ of your fortune, and then I can pay my debts. But B answered, lend me the $\frac{1}{6}$ of your fortune, and I can pay my debts. What was the fortune of each?

Ans. A \$900; B \$2400.

23. A wine merchant has two kinds of wine, one at 5*s.* a gallon, and the other at 12*s.*; of which he wishes to make a mixture of 20 gallons that shall be worth 8*s.* a gallon. How many gallons of each sort must he use?

Ans. $8\frac{4}{7}$ gallons of that at 12*s.*; $11\frac{3}{7}$ of that at 5*s.*

SECOND METHOD OF EXTERMINATION.

§126. In each of the preceding questions, we first found the value of *one* of the unknown quantities; and then substituted that value for that unknown quantity in one of the equations, in order to find the value of the other unknown quantity. This mode of operating furnishes a hint that leads us to another method of extermination.

Let us take the first question in the last section [p. 88,] in which we have the equations, $\begin{Bmatrix} x+y=20 \\ x-y=12 \end{Bmatrix}$

The last part of our operation was to substitute the value of x for x itself, in one of the equations. It is evident that we could make this substitution just as well if the value of x was a *literal* quantity, instead of 16. Thus, supposing

x to be equal to $\frac{y}{2}$; then substituting it for x, the first equation would be $\frac{y}{2}+y=20$.

§127. Let us therefore transpose the first equation to find what x will equal, just as if we knew the value of y. We shall find that $x=20-y$. And then in the second equation, we will use the value of x instead of x itself.

$$\text{Thus, } 20-y-y=12.$$

Transposing and uniting, $-2y=-8 \therefore y=4$; which was our answer by the first method. Then x will be found by substituting 4 for y. Whence we derive

RULE II. TO EXTERMINATE AN UNKNOWN QUANTITY.

§128. *By one of the equations, find the value of one of the unknown quantities, as if the other were known; and then, in the other equation, substitute this value for the unknown quantity itself.*

EQUATIONS.—SECTION 13.

1. There are two numbers whose sum is 100; and three times the less taken from twice the greater, leaves 150 remainder. What are those numbers?

Let $x=$ greater.
$y=$ less.
$2x-3y=$ the required subtraction.

1. By the first condition, $x+y=100$
2. By the second, $2x-3y=150$
3. Transposing the 1st, $x=100-y$
4. Multiplying the 3d by 2, $2x \quad 200-2y$
5. Substituting $200-2y$ for x in the 2d, $200-2y-3y=150$
6. Transposing and uniting, $-5y=-50 \therefore y=10$

7. Substituting 10 for y in the 1st, $x+10=100$
8. Transposing and uniting, $x=90$

Ans. Greater 90; Less 10.

2. The ages of a father and his son amounted to 140 years; and the age of the father was to the age of the son as 3 to 2. What were their ages?

Let $x=$ age of the father.
$y=$ age of the son.

1. By the first condition, $x+y=140$
2. By the second, $y=\frac{2x}{3}$
3. Transposing the 1st, $y=140-x$
4. Substituting $140-x$ in the 2d, $140-x=\frac{2x}{3}$
5. Multiplying by 3, $420-3x=2x$
6. Transposing and dividing, $x=84$
7. Substituting 84 for x in 3d, $y=140-84=56$.

Ans. Father 84 years; Son 56.

3. Find two numbers, such that $\frac{1}{3}$ of the first and $\frac{1}{4}$ of the second shall be 87; and $\frac{1}{5}$ of the first and $\frac{1}{6}$ of the second shall be 55. Ans. 135; and 168.

4. A says to B, give me 100 of your dollars, and I shall have as much as you. B replies, give me 100 of your dollars and I shall have twice as much as you. How many dollars has each? Ans. A \$500; B 700.

5. Two servants went to market. A laid out as much above 4 shillings, as B did under 6; and the sum spent by A was to that spent by B, as 7 to 8. How much did each lay out? Ans. A, 4*s*. 8*d*.; B, 5*s*. 4*d*

6. Find two numbers in the proportion of 2 to 1, so that if 4 be added to each, their two sums shall be in proportion of 3 to 2. Ans. 8 and 4.

7. A and B owned 9800 acres of western land. A sells $\frac{1}{6}$ of his, and B sells $\frac{1}{5}$ of his; and they then have just as much as each other. How many acres had each?

Ans. A 4800; B 5000.

8. A son asking his father how old he was, received the following reply. My age, says the father, 7 years ago was four times as great as yours at that time; but 7 years hence, if you and I live, my age will be only double of yours. What was the age of each?

Ans. Father's 35 years; Son's 14 years.

9. The weight of the head of Goliah's spear was less by one pound than $\frac{1}{8}$ the weight of his coat of mail; and both together weighed 17 pounds less than ten times the spear's head. What was the weight of each?

Ans. Coat, 208 pounds; Spear's head, 25 pounds.

10. A market woman bought eggs, some at the rate of 2 for a cent, and some at the rate of 3 for 2 cents, to the amount of 65 cents. She afterwards sold them all for 120 cents, thereby gaining half a cent on each egg. How many of each kind did she buy?

Ans. 50 of the first kind; 60 of the other kind.

11. Says A to B, $\frac{1}{3}$ of the difference of our money is equal to yours; and if you give me \$2, I shall have five times as much as you. How much has each?

Ans. A \$48; B \$12.

12. A and B possess together property to the amount of \$5700. If A's property were worth three times as much as it is, and B's five times as much as it is, then they both would be worth \$23,500. What is the worth of each?

Ans. A \$2500; B \$3200.

13. A gentleman has two silver cups, and a cover adapted to each which is worth \$20. If the cover be put upon the first cup, its value will be twice that of the second; but if it be put upon the second, its value will be three times that of the first. What is the value of each cup?

Ans. First cup, \$12; second, \$10.

14. Two men driving their sheep to market, A says to B, give me one of your sheep and I shall have as many as you. B says to A, give me one of your sheep, and I shall have twice as many as you. How many had each?

Ans. A, 5 sheep; B, 7.

THIRD METHOD OF EXTERMINATION.

§129. The method of substitution as explained in the last chapter, may be modified a little. We will show how, by using question 1st, in the last section of equations.

The two equations were $\begin{cases} x+y=100 \\ 2x-3y=150 \end{cases}$

We transposed the 1st; thus, $x=100-y$

Now, before we substitute the value of x for x itself in the second equation, we will transpose the second equation so as to make x stand alone; thus, $2x=150+3y$.

Then substitute the value of x as found before by the first equation, $200-2y=150+3y$ with which we may proceed as before.

§130. Before we make the substitution after transposing, it is generally best to find the value of x *alone* in the second equation. Thus,

Given $\left\{\begin{matrix} 2x+3y=23 \\ 5x-2y=10 \end{matrix}\right\}$ to find x and y.

Transposing and dividing the 1st, $x=\frac{23-3y}{2}$

Transposing and dividing the 2d, $x=\frac{10+2y}{5}$.

Now, as it is evident that *things which are equal to the same, are equal to one another;* one value of x is equal to the other value of x; thus,

$$\frac{23-3y}{2}=\frac{10+2y}{5}$$

Destroying the fractions, $115-15y=20+4y$

Transposing, uniting, and dividing, $y=5$

By substituting the value of y in one of the equations, we find $x=4$. Whence we derive

RULE III. TO EXTERMINATE AN UNKNOWN QUANTITY.

§131. *Find by each of the equations, the value of that unknown quantity which is the least involved; and then form a new equation by making one of these values equal to the other.*

EQUATIONS.—SECTION 14.

1. Divide $60 between A and B, so that the difference between A's share and 31, may be to the difference between 31 and B's share, as 6 to 7.

Let $x=$ A's share; and $y=$ B's.

1. By the first condition, $x+y=60$
2. By the second, $x-31:31-y::6:7$
3. Multiplying extremes and means, $7x-217=186-6y$
4. Transposing the 1st, $x=60-y$
5. Transposing and uniting the 3d, $7x=403-6y$
6. Multiplying the 4th, $7x=420-7y$
7. Making 5th and 6th equal, $403-6y=420-7y$
8. Transposing and uniting, $y=17$
9. Substituting 17 in the 4th, $x=60-17=43$

Ans. A's share $43; B's $17.

2. There is a fraction such that if 1 is added to the numerator, its value will be $\frac{1}{3}$; but if 1 be added to the denominator, its value will be $\frac{1}{4}$. What is that fraction?

Let $x=$ numerator; and $y=$ denominator.

1. The fraction will be, $\frac{x}{y}$
2. By the first condition, $\frac{x+1}{y}=\frac{1}{3}$
3. By the second, $\frac{x}{y+1}=\frac{1}{4}$
4. Multiplying the 2d by y, and by 3, $3x+3=y$
5. Multiplying the 3d by $y+1$, and by 4, $4x=y+1$
6. Transposing and dividing the 5th, $4x-1=y$
7. Making 4th and 6th equal, $3x+3=4x-1$
8. Transposing and uniting, $-x=-4$
9. Substituting the value of $3x$ in the 4th, $15=y$.

Ans. $\frac{4}{15}$.

3. What two numbers are those, whose difference is 4 and 5 times the greater is to 6 times the less, as 5 to 4?

Ans. 8 and 12.

4. There is a certain number, consisting of two places of figures, which is equal to 4 times the sum of its digits; and if 18 be added to it, the digits will be inverted. What is that number?

Let $x=$ first digit or *tens;* and $y=$ the units.

$10x+y=$ the number.

$4x+4y=$ four times the sum of digits.

$10x+y+18,=$ when 18 is added.

$10y+x,=$ when the digits are inverted.

By the first condition, $10x+y=4x+4y$

By the second, $10x+y+18=10y+x$

Ans. 24.

5. There is a certain number consisting of two figures; and if 2 be added to the sum of its digits, the amount will be three times the first digit; and if 18 be added to the number, the digits will be inverted. What is the number?

Ans. 46.

6. A person has two snuff-boxes and \$8. If he puts the 8 dollars into the first, then it is half as valuable as the other. But if he puts the 8 dollars into the second, then the second is worth three times as much as the first. What is the value of each? Ans. First \$24; second \$64.

7. A gentleman has two horses and a chaise. The first horse is worth \$180. If the first horse be harnessed to the chaise, they will together be worth twice as much as the second horse; but if the second horse be harnessed, the horse and chaise will be worth twice and one half the value of the first. What is the value of the second horse, and of the chaise? Ans. Horse \$210; Chaise \$240.

8. There is a certain number consisting of two digits. The sum of these digits is 5; and if 9 be added to the number itself, the digits will be inverted. What is the number? Ans. 23.

9. There are two numbers such that $\frac{1}{2}$ of the greater added to $\frac{1}{3}$ of the less, will equal 13; and if $\frac{1}{2}$ of the less be taken from $\frac{1}{3}$ of the greater, the remainder is nothing. What are the numbers? Ans. 18 and 12.

10. There is a number consisting of two figures. If the number be divided by the sum of the figures, the quotient will be 4; but if the number made by inverting the figures, be divided by 1 more than their sum, the quotient will be 6. What is the number? Ans. 24.

11. There are two numbers such that the less is to the

greater as 2 to 5; and the product made by multiplying the two numbers together is equal to ten times their sum. What are the numbers?

Let $x=$ the less; and $y=$ the greater.

1. By the first condition, $x=\frac{2y}{5}$
2. By the second, $xy=10x+10y$

NOTE.—If we wish to multiply y by 4, we put 4 immediately before the y as a co-efficient; and in the same way, if we multiply y by x, we make x the co-efficient of y.

3. Destroying the fraction of the 1st, $5x=2y$
4. Multiplying by 2, $10x=4y$
5. Substituting $4y$ for $10x$ in 2d, $xy=4y+10y$
6. Dividing by y, $x=4+10=14.$

NOTE.—When we divide $4y$ by 4, we do it by taking away 4; when we divide $10y$ by 10, we do it by taking away the 10. In the same manner, we divide by y, in taking away the y.

7. Substituting 14 for x in the 3d, $70=2y \therefore y=35.$

Ans. 14 and 35.

12. There are two numbers, whose sum is the $\frac{1}{6}$ part of their product; and the greater is to the less as 3 to 2. What are those numbers? Ans. 15 and 10.

EQUATIONS WITH SEVERAL UNKNOWN QUANTITIES.

§132. When there are three or more unknown quantities, they are exterminated one after another by the same methods which are used for two unknown quantities.

§133. But it must always be the case that there are as many independent equations as there are unknown quantities. See §117.

EXAMPLES.

1. Given the equations $\left\{\begin{array}{l} x+y+\ z=\ 9 \\ x+2y+3z=16 \\ x-y-2z=-3 \end{array}\right\}$ to find x, y, and z

4. Subtracting 3d from 1st, $2y+3z=12$
5. Subtracting 1st from 2d, $y+2z=7$
6. Multiplying 5th by 2, $2y+4z=14$
7. Subtracting 4th from 6th, $z=2$
8. Substituting val. of z in 5th, $y=3$
9. Substituting in the 1st, $x=4$.

2. Given $\left\{\begin{array}{l} x+y\ +z\ =29 \\ x+2y+3z=62 \\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=10 \end{array}\right\}$ to find x, y, and z.

4. Destroying fractions in 3d, $6x+4y+3z=120$
5. Subtracting 1st from 2d, $y+2z=33$
6. Multiplying 1st by 6, $6x+6y+6z=174$
7. Subtracting 4th from 6th, $2y+3z=54$
8. Multiplying 5th by 2, $2y+4z=66$
9. Subtracting 7th from 8th, $z=12$

Whence by substitution, $y=9$; and $x=8$.

3. Given $x+y+z=7$; $2x-y-3z=3$; and $5x-3y+5z=19$; to find x, y, z. Ans. $x=4$, $y=2$, $z=1$.

4. Given $x-y-z=5$; $3x+4y+5z=52$; and $5x-4y-3z=32$; to find x, y, and z. Ans. $x=10$, $y=3$, $z=2$.

5. Given $7x+5y+2z=79$; $8x+7y+9z=122$; and $x+4y+5z=55$; to find the values of x, y, and z.

Ans. $x=4$, $y=9$, $z=3$.

6. Given $x+y+z=13$; $x+y+u=17$; $x+z+u=18$; and $y+z+u=21$; to find the values of x, y, z, and u.

Ans. $x=2$, $y=5$, $z=6$, $u=10$.

PART II.

LITERAL ALGEBRA.

GENERAL PRINCIPLES.

§134. The Algebraical operations which we have hitherto treated of, belong to that part of the science which was known to the ancients, and which was in use till about A. D. 1600. About that time, Franciscus Vieta, a Frenchman, introduced the general use of letters into Algebra, (denoting the known quantities in a problem by consonants, and the unknown ones by vowels.)

§135. This improvement gave a new aspect to the science. So that now the object of it is, to afford means for discovering general rules for the resolution of all questions that can be proposed relative to quantities.

§136. Operations with numbers cannot furnish general rules; because they show only the results without denoting how they were obtained. Thus, 12 may be the result, either of multiplying 3 by 4, or adding 5 to 7, or of subtracting 8 from 20, or of dividing 48 by 4, &c. &c. In order that a result should furnish a rule, it should not depend upon the particular values of the quantities which we operate with, but rather upon the nature of the question; and should also be the same for every question of the same kind.

§137. Algebra enables us to discover rules, because it is rather the representation of arithmetical results, than the results themselves.

§138. In the first place, it represents the *quantities* under consideration by *general signs;* that is, the letters of the alphabet; which may stand for whatever number we choose, and even for as many different numbers as we choose, one after another.

§139. In the next place, by different methods of combining these representatives of numbers, it represents the *operations* which are made with them; and this is done in such a manner that the quantities still retain their identity.

§160. And further, that these operations may be facilitated and rendered perspicuous, many of them are represented by general characters called signs. The following are the most common.

The sign $+$ (*plus*) represents addition.

The sign — (*minus*) represents subtraction.

The sign $\times$ (*multiplied by*) represents multiplication.

The sign . is sometimes put between two literal quantities instead of $\times$; as $a.b$. But more generally, in Algebra the letters are joined together, to represent their product.

The sign $\div$ (*divided by*) represents the division of the quantity before it by the quantity which follows it.

Division is more generally denoted in Algebra by writing the divisor under the dividend in the form of a fraction.

The sign $=$ (*equals*) denotes that the whole quantity on the left of it, is equal to the whole quantity on the right.

The sign $\pm$ or $\mp$ (*plus or minus* and *minus or plus*,) shows that either by addition or subtraction, the effect will be the same.

The sign $\backsim$ or $\sim$ denotes the difference between two quantities.

The sign $>$ or $<$ (*greater than* or *less than*) denotes that that quantity towards which it opens is greater than the other.

The sign $\overline{\quad\quad}$ (a *vinculum*,) denotes that all which is put under it, is to be used as one term.

The ( ) *parenthesis* is frequently used instead of the vinculum.

The sign ∞ (*infinity*) denotes a quantity that is infinitely large, or a quantity so great that it may be considered larger than any supposable quantity.

The *cipher* 0 is sometimes used to represent a quantity that is less than any quantity that may be mentioned. This is always the case when the cipher is used as a *denominator* of a fraction.

The radical sign $\surd$ denotes the *root* of the following quantity. When unaccompanied by a figure, it represents the *square* root. But when a figure is put over it, that figure expresses the root that is designed.

A *co-efficient* is a number put immediately before a letter, as, $2b$.

In such a case, the co-efficient is called a *numeral* co-efficient. Sometimes when one *letter* has been multiplied into another, the first written one is called a *literal* co-efficient; as, ab.

An *exponent* or *index* is a small figure placed a little over and a little to the right of a quantity; as, a^2.

An exponent may be either positive, negative, or fractional, as, a^2, a^{-2}, $a^{\frac{1}{2}}$.

The sign : : represents proportion, and : denotes the ratio of two numbers.

The sign $\propto$ denotes a general proportion.

A *simple quantity* is that which is represented by one term.

A *compound quantity* is that which is represented by two or more terms.

A quantity when represented by one term is sometimes called a *nomial;* with two terms it is called a *binomial;* with three it is called a *trinomial;* with many terms it is called a *multinomial*, or *polynomial.*

§141. The algebraical mode of designation is of the greatest use in universal arithmetic; as every conclusion, and indeed every step by which it is obtained, becomes a general rule for performing every possible operation of the kind.

Example 1. Suppose we wish to divide $660 between a son and a daughter, so that the daughter shall have twice as much as the son. What must we give to each?

Now, a question similar to this, and with the same numbers was solved in the First section of Equations, on page 26. We will solve this in the same manner, with the exception of using a instead of 660.

Stating the question, $x=$ What the son has.

$2x=$ What the daughter has.

Both together have $x+2x$; also they have a dollars.

Forming the equation, $x+2x=a$

Uniting terms, $3x=a$

Dividing by 3, $x=\frac{a}{3}$

Here we find that the son's share is $\frac{1}{3}$ of a, which at this time stands for $660.

But it is very plain that we would solve the question in the very same manner, if the sum were $240. And in that case a would stand for 240; and the son's share of it would be $80, and the daughter's share, $160.

In the same manner, if the sum were \$360; then a would stand for \$360, and the son's share would be $\frac{1}{3}$ of 360; that is, \$120. And in the same manner we may make a represent any sum; and still the son's share of it would be $\frac{1}{3}$ of it. Hence, this substitution of a letter for a number, is called *generalizing the operation.*

We see that this result has given us a *general rule* for dividing any sum between two, so that one of them shall have twice as much as the other. The rule is, *The least share shall be one-third of the sum; and the greatest share, two-thirds of it.*

§142. In the same manner, we may generalize all the questions in the First Section of Equations. That is, with each question we may find a rule by which any other question like it, may be answered with fewer figures than the Algebraic operation required. But, as such sums are not apt to occur, there will be no practical use of finding rules for them. Notwithstanding, as the exercise may be instructive, the teacher may require his pupils to go through them if he sees fit.

Example 2. What number is that, which, with 5 added to it, will be equal to 40?

This is the first problem in section 2, which we will generalize; using a for 40, and b for 5.

Stating the question,	$x=$ the number.
	$x+b=$ after adding.
Forming the equation,	$x+b=a$
Transposing b,	$x=a-b$

We see that the answer is found by subtracting the 5 from the 40. Thus, $40-5=35$.

The third question is similar to it; and in that, a represents 23, and b represents 9. The answer is $23-9=14$.

Example 3. Divide \$17 between two persons, so that one may have \$4 more than the other. [Prob. 4. Sect. 2.]

Represent 17 by a, and 4 by b.

Forming the equation, $x+x+b=a$

Transposing b, $x+x=a-b$

Uniting terms, $2x=a-b$

Dividing by 2, $x=\dfrac{a-b}{2}$

The answer is found by subtracting the difference or 4, from the whole sum, and then dividing by 2.

$$\text{Thus, } \frac{17-4}{2}=\frac{13}{2}=6\tfrac{1}{2}.$$

And this is the rule for all similar sums.

The 5th question is similar to it; and in that, a represents 55, and b represents the difference or 7. It is found by the rule just shown.

$$\text{Thus, } \frac{55-7}{2}=\frac{48}{2}=24.$$

Perform the 7th and 10th by the same rule.

§143. As this rule is of some importance, it will be well to remember it. *If, from a number to be divided into two parts, we subtract the difference of those parts, half the remainder will be equal to the smaller part.*

Example 4. In the same questions, let us take x for the greatest share. Then $x-b=$ the less.

Forming the equation, $x+x-b=a$

Transposing and uniting, $2x=a+b$

Dividing by 2, $x=\dfrac{a+b}{2}$

§144. Here we have another rule. *If, to a number to be divided into two parts, we add the difference between those parts, half the sum will be equal to the greater part.*

Find the greater part in questions 4, 5, 7, and 10, by this rule.

§145. The mere letters in the answer of an algebraical operation are called a *formula*. They are not called a *rule*, until they are turned into common language.

Example 5. The learner may now generalize question 6; and by the formula that he obtains, he may find the answers in questions 8, 9, and 11. The two differences will be b and c.

We have said that in Algebra the arithmetical operations on numbers are only *represented* by different methods of combining the signs that stand for those quantities. And now, although we have shown in our progress thus far, what some of those methods are, it may be well to review them a little.

ADDITION AND SUBTRACTION OF ALGEBRAICAL QUANTITIES.

§146. One algebraical quantity is added to another by writing one quantity after the other, taking care to put the sign plus $+$ between them. Thus, $a+f-c$ is added to $d-e+b$, so as to make $d-e+b+a+f-c$. Or, as it is easier to read the letters in their alphabetical order, their sum may be written $a+b-c+d-e+f$.

§147. One algebraical quantity is subtracted from another, by changing the sign or signs of the quantity which is to be subtracted, and then writing that quantity after the other. Thus, $a+h-y$ is subtracted from $b-x+c$, by first

making it $-a-h+y$, and then writing the whole quantity, $b-x+c-a-h+y$; or, $b-a+c-h+y-x$.

§148. When quantities that have been made by addition or subtraction, have like terms in them, they may be reduced to *smaller expressions*, by uniting the like terms. This is done by putting into one part, all those which have the sign $+$, and into another part, all those which have the sign $-$; then subtract the least result from the greatest, and give to the remainder the sign that belonged to the greatest result.

§149. In uniting the terms of compound numbers, we consider the *literal part* of the term as a unit; thus, $2a$ and $3a$, are regarded as 2 units and 3 units of a particular kind, which when put together, make 5 units of that kind. Now, we have seen, §139, that the *co-efficient* of a quantity may also be literal; as in ba, ca, &c. In such cases, the *whole term* ba or ca becomes the unit, each of a different kind; and of course are not like quantities and cannot be united.

§150. But if there are several similar units of this kind, they may be united by the general rule. Thus, $ba-ca+ba+ca+ba+ca$, can be united into, $3ba+ca$. $ax-bx+ax+2bx-3ax+bx$, are equal to, $-ax+2bx$; or $2bx-ax$.

§151. Again, we have seen, §65, that several quantities are sometimes united by a vinculum. In such cases all that is embraced by the vinculum, is regarded as a unit of that kind; and may have a co-efficient. Thus, in the expressions, $3\times\overline{a-b+x}$, and $5(x+ax-y)$, $a-b+x$ is a quantity taken 3 times, and $x+ax-y$ is a quantity taken 5 times. Like quantities of this kind can be united; thus, $2(ay-bx+x)+5(ay-bx+x)=7(ay-bx+x)$.

§152. In uniting terms, great care must be taken that the literal part be entirely alike. Thus, $2bx+3cx$, cannot be united. Neither can $3y-2ay$; nor, $6(a+bx)+2(ax+bx)$; nor, $3.\overline{ay-by}+2.\overline{ay^2-by}$, nor, $4(ax-bx)-2(ax+bx)$; neither in any other case where there is the least difference in any part but the leading co-efficient.

EXAMPLES.

Unite the following quantities.

1. $3ax-2y+4x-5y+ax-3y$. Ans. $8ax-10y$.
2. $3x+ay-2x-ay+4x+3ay-2x+4ay$. Ans. $3x+7ay$.
3. $4ax-y+3ay-2-2ax+ay-7y+8+2ay+y$. Ans. $2ax+6ay-7y+6$.
4. $ax-ay^2-3ay+5ax-2ay+7ay-4ax-8ay^2$. Ans. $2ax+2ay-9ay^2$.
5. $3(a-y)+4(a-y)+2(a-y)+7(a-y)$. Ans. $16(a-y)$.
6. $-4(a+b)+3(a+b)-2(a+b)+7(a+b.)$ Ans. $4.\overline{a+b}$.
7. $2(ab+x)+3(ax+b)-4(x-y)-2(ab+x)$ Ans. $3(ax+b)-4(x-y)$.
8. $7y-4(a+b)+6y+2y+2(a+b)+(a+b)+y-3(a+b.)$ Ans. $16y-4(a+b)$.
9. $x^2+ax-ab+ab-x^3+xy+ax+xy-4ab+x^2+x^3-x+xy+xy+ax$. Ans. $2x^2+3ax-4ab+4xy-x$.
10. $x+12-ax+y-(48-x-ax+3y)$. Ans. $2x-36-2y$.

11. $ab-4xy-a-x^2-(2xy-b+14x+x^2.)$

Ans. $ab-6xy-a+b-14x-2x^2$.

12. $3(x+y)+(4.\overline{x+y})$. Ans. $7(x+y)$.

13. $2(a-b)-x-(3.\overline{a+b}-x^2)$. Ans. $x^2-(a+b)-x$.

14. From $4.\overline{a+b}$, take $\overline{a+b}-3.\overline{x-y}$.

Ans. $3.\overline{a+b}+3.\overline{x-y}$.

15. $a+b-(2a-3b)-(5a+7b)-(-13a+2b)$.

Ans. $7a-5b$.

16. $37a-5x-(3a-2b-5c)-(6a-4b+3h)$.

Ans. $28a+6b-5x+5c-3h$.

EXERCISES IN EQUATIONS.

§153. The learner may now generalize the problems in Sections 3 and 4 of Equations, pages 35, and 40. This he can easily do, if he takes care to make one of the first letters of the alphabet stand for each numeral quantity. When the same numeral quantity occurs more than once in the same question, the same letter must stand for it each time.

MULTIPLICATION OF ALGEBRAICAL QUANTITIES.

§154. We have shown, §29, that a literal quantity may be multiplied by writing the multiplier before that quantity. This is the case whether the multiplier is numeral or literal. Thus, a times x, is written ax; $b\times c=bc$. In the same manner, a times bc becomes abc; and f times abc becomes $fabc$. As $f\times abc$ is the same as $abc\times f$, we see that it is of no consequence what order we make of the letters in the

product. $abcd = acdb = cadb$, &c. But it is generally more convenient to follow the order of the alphabet.

§155. Therefore, to multiply one simple quantity by another, write the quantities one after another, without any sign between them. Thus, abx times $cfy = abxcfy$; $5ax$ times $cdf = 5acdfx$. But, *if there are more than one* NUMERAL *co-efficient, those co-efficients must be multiplied as in arithmetic*, and placed before the product of the literal quantities. Thus, $3a \times 2x = 3.2ax = 6ax$. $2bc \times 5rs = 10bcrs$.

EXAMPLES.

1. Multiply a by b. Ans. ab.
2. Multiply ab by c. Ans. abc.
3. Multiply ab by cd. Ans. $abcd$.
4. Multiply $2acx$ by by. Ans. $2abcyx$.
5. Multiply $3brs$ by $2mnx$. Ans. $6bmnrsx$.
6. Multiply $2adn$ by $5cmx$.
7. Multiply $3rsyop$ by $4antx$.
8. Multiply $2abcx$ by $8abrx$.

§156. By the foregoing principle, $a \times a = aa$. Now, as in Algebra the same factor is often found two or more times, Stifelius adopted a method for shortening such expressions,* in which he has ever since been followed. The method is this; *when the same letter enters as a factor two or more times into any quantity, we write the factor but once, and put at the right of it and a little raised, a figure denoting how many times it has been multiplied.* Thus, aa is written a^2; bbb is written b^3; $xxxx$ is written x^4; $aabbbyyyy$ is written $a^2 b^3 y^4$.

* A. D. 1554.

§157. Mathematicians are accustomed to call aa or a^2, the second power of a, or, *a-second power;* a^3, is called *a-third power*, &c.

§158. The figure that denotes the power of any quantity is called the *exponent* or *index* of that quantity.

§159. All quantities are said to have an exponent, either expressed or understood. Thus, a is the same as a^1; $b=b^1$; &c. The written exponent affects no letter except the one over which it is written; unless it is denoted by a vinculum.

§160. Great care must be taken by the pupil not to confound the *co-efficient* with the *exponent;* as their effects are entirely different. The co-efficient shows addition, the exponent denotes multiplication. For example, if $a=5$, then $3a=5+5+5=15$; but $a^3=5\times5\times5=125$.

§161. We have seen that $a^2=aa$; and that $a^3=aaa$; now $aa\times aaa=aaaaa$. So we see that $a^2\times a^3=a^5$. Hence we establish the rule that *when both multiplier and multiplicand are denoted by the same letter, their product is found by adding their exponents.* $x^2\times x^4=x^6$, $y^3\times y^2=y^5$; &c.

EXAMPLES.

9. Multiply $2amn$ by a. Ans. $2a^2mn$.
10. Multiply $3abcx$ by $4ax$. Ans. $12a^2bcx^2$.
11. Multiply $5bcmn$ by $3bc$. Ans. $15b^2c^2mn$.
12. Multiply $6a^2xy$ by $4ax^2y$. Ans. $24a^3x^3y^2$.
13. Multiply $4a^2bcx$ by a^4b^2c. Ans. $4a^6b^3c^2x$.
14. Multiply $3a^3m^2$ by a^4m^3n.
15. Multiply $5a^7x^6$ by $4a^3x^4$.
16. Multiply $2m^5r^2a^2x$ by $9a^7x^4rn^2$.

17. Multiply $3b^4c^2n^5$ by $8a^3d^2mn^3$.

18. Multiply $7a^3y^2x^4$ by $12a^5b^2x$.

19. Multiply $9m^7x^2a^3$ by $7a^2b^3m$.

20. Multiply $12a^2m^6nx^5$ by $18b^3cn^8y$.

§162. When one of the factors is a compound quantity, (§61 and 62,) we multiply each term by the other factor, and set down their products, each with their proper sign. In doing this, we generally begin at the *left*; thus,

	$a+bc$		$ab-c.$
Multiplied by	x	Multiplied by	a
	$ax+bcx$		$a^2b-ac.$

21. Multiply $ax+bx$ by xy. Ans. ax^2y+bx^2y.

22. Multiply $2a^2bc-rxy^3$ by $2ay$. Ans. $4a^3bcy-2arxy^4$.

23. Multiply $7ay+1$ by $3r$. Ans. $21ary+3r$.

24. Multiply $2xy+ab+c^2$ by $2ax$.

25. Multiply $5-7x+2a^3b$ by $4ac^2y$.

26. Multiply $3m^2n-3rp+2sx^3$ by $6a^2ba$.

27. Multiply $2en-4an+5$ by $12a^2x^5$.

28. Multiply $4y+y^2$ by $2xy$.

29. Multiply $3a-4b+4$ by $5y$.

30. Multiply $6ax^2-a^2-1$ by $3ax^2$.

31. Multiply $4+3a-x^2$ by ay.

32. Multiply $2a+5b^2+3c-5e$ by $3a^2$.

33. Multiply $7b^2-4+a^2x-x^2$ by $4a^2x$.

34. Multiply $6x+7a-axy-2y$ by $3bx^2$.

§163. As $(a+b)\times x=ax+bx$; we know that $x\times(a+b)$ also $=ax+bx$. Whence we see that if we wish to multiply x by $a+b$; we first find the product of a times x, and then add to it the product of b times x. And generally,

when the multiplier is composed of several terms, the product is made up of the sum of the products of the multiplicand by each term of the multiplier. Thus, $(x+y)\times(a+b)$

$$=\left\{\begin{array}{l} x+y \\ a \\ \hline ax+ay \end{array}\right\} \text{ and } \left\{\begin{array}{l} x+y \\ b \\ \hline bx+by \end{array}\right\} \text{ or } \left\{\begin{array}{l} x+y \\ a+b \\ \hline ax+ay+bx+by. \end{array}\right.$$

EXAMPLES.

35. Multiply $2a+3b$ by $4a+5b$.

$$\begin{array}{l} 2a+3b \\ 4a+5b \\ \hline 8a^2+12ab \\ \quad\ +10ab+15b^2 \\ \hline 8a^2+22ab+15b^2 \end{array}$$

36. Multiply $a+b$ by $a+b$. Ans. $a^2+2ab+b^2$.

37. Multiply $a+b-c$ by a^2+b.
Ans. $a^3+a^2b-a^2c+ab+b^2-bc$.

38. Multiply $x^2+2xy+y^2$ by $x+y$.
Ans. $x^3+3x^2y+3xy^2+y^3$.

39. Multiply $a-x$ by $2a+3x$. Ans. $2a^2+ax-3x^2$.

40. Multiply $x-4$ by $x+8$. Ans. $x^2+4x-32$.

41. Multiply $x-y$ by $x+y$. Ans. x^2-y^2.

42. Multiply $4a^2-3xy^2+7ax-ay$ by $2a^2+x^2y$.

43. Multiply $a^2bc^3+3a^3c^2-5b^2c$ by ax^2+yx^2.

44. Multiply $m^3a^2+2n^2b^3-1-m^4n$ by $2m^3b^2+4a^3n^4$.

45. Multiply $8a^2x^3y^4-2a^4x^3y^2+a^5y^5$ by $3a^2m+2ny^4$.

§164. As $(a-x)\times b=ab-xb$; we also know that $b\times(a-x)=ab-xb$. Therefore to find $a-x$ times b, we first find the product of a times b, and then subtract from it x times b. This principle is very plain in figures. Thus, suppose we have 7 times 8, and wished only 4 times 8; if we take

3 times 8 away from the 7 times 8, we should obtain 4 times 8; as 56—24=32. Whence we have the general rule, *if there is a negative quantity in the multiplier; the product of that quantity and the multiplicand must be subtracted from the product of the positive quantities and the multiplicand.* Thus, $(a+b)\times(c-d)$.

$$=\left\{\begin{array}{l} a+b \\ c \\ \hline ac+bc \end{array}\right\} \text{ minus } \left\{\begin{array}{l} a+b \\ d \\ \hline ad+bd \end{array}\right\} \text{ or } \begin{array}{l} a+b \\ c-d \\ \hline ac+bc-ad-bd. \end{array}$$

Here we see that the product of d into $a+b$ is subtracted from the product of c into $a+b$; and therefore the signs of $ad+bd$ are changed to $-ad-bd$.

§165. By examining the answer of this last example, we shall observe a principle which will enable us to be more rapid in the multiplying operation. It is this; *when we multiply a + term by a + term, the product in the answer is a + term; and when we multiply a + term by a — term, the product in the final answer is a — term.* It will be well for the pupil to explain this.

By understanding this principle, we are able to set the final answer down at first; as we have only time to remember that,

§166. When the signs are *alike,* the product is +.

§167. When the signs are *unlike,* the product is —.

EXAMPLES.

46. Multiply $a+b$ by $a-b$.

$$\begin{array}{lll} a & +b & \\ a & -b & \\ \hline a^2 & +ab & \\ & -ab & -b^2 \\ \hline a^2 & & \;\;b^2 \end{array} \qquad \text{Ans. } a^2-b^2.$$

47. Multiply $x+8z$ by $3x^2-7xz$.

Ans. $3x^3+17x^2z-56xz^2$.

48. Multiply x^2+xy+y^2 by $x-y$. Ans. x^3-y^3.

49. Multiply $2x+3y$ by $3x-4y$. Ans. $6x^2+xy-12y^2$.

50. Multiply $b^3+b^2x+bx^2+x^3$ by $b-x$. Ans. b^4-x^4.

51. Multiply $a-b$ by $c-d$.

$$\left.\begin{array}{l} a-b \\ \underline{c\qquad} \\ ac-bc \end{array}\right\} \text{ minus } \left\{\begin{array}{l} a-b \\ \underline{d\qquad} \\ ad-bd \end{array}\right\} \text{ or } \begin{array}{l} a-b \\ \underline{c-d\qquad\qquad} \\ ac-bc-ad+bd. \end{array}$$

Here we see that in subtracting d times $a-b$, we change the signs of $ad-bd$ to $-ad+bd$.

§168. Whence we learn,

+ multiplied by +, produces +
+ multiplied by —, produces —
— multiplied by +, produces —
— multiplied by —, produces +.

And by remembering this we can always set down the *final answer* at first.

52. Multiply $a-x$ by $a-x$.

$$\begin{array}{r} a-x \\ \underline{a-x } \\ a^2-ax \\ \underline{-ax+x^2} \\ a^2-2ax+x^2. \end{array}$$

53. Multiply $2x-3a$ by $4x-5a$.

Ans. $8x^2-22ax+15a^2$.

54. Multiply $2a-5y$ by $a-2y$. Ans. $2a^2-9ay+10y^2$.

55. Multiply a^2+ac-c^2 by $a-c$. Ans. $a^3-2ac^2+c^3$.

56. Multiply $a+b-d$ by $a-b$.

Ans. a^2-ad-b^2+bd.

57. Multiply $4x-5a-2b$ by $3x-2a+5b$.

Ans. $12x^2-23ax+14bx+10a^2-21ab-10b^2$.

58. Multiply $x^3-y^3-z^3$ by $x-y-z$.

59. Multiply $6+xy-a^2-my^3$ by $a^2-3x^3+y^4$.

60. Multiply $2a^3b-3ac^2+4b^3c^2-1$

by $2a^3c^2-5b^2c-8a^2$.

§169. In order to facilitate the practice of multiplication, it is best to observe the following method. First, determine the sign, then the co-efficient, then the letters in their order, and then the exponents.

General Properties of Numbers.

§170. We have before stated that algebraical operations, §137, by reason of the quantities themselves being retained in their original value, do show us, in their results, important general principles. We will here make a few multiplications of some quantities, whose results show us some remarkable general properties of numbers. These properties the pupil should remember, as they are of frequent use in the subsequent parts of this study.

§171. Suppose we have two numbers, a and b, of which a is the greatest. Then their sum $=a+b$; and their difference $=a-b$. Then

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ -ab-b^2 \\ \hline a^2-b^2 \end{array}$$

By this operation, we find the general property of numbers which it would be difficult to find by any arithmetical operation. It is that, *if we multiply the sum of two numbers by their difference, the product will be the difference of the squares of those numbers.*

§172. Again, take the same quantities, and multiply their sum, by their sum.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

By this operation we find the following general property. *The square of the sum of two numbers is equal to the square of the first number, plus twice the product of the two numbers, plus the square of the last number.*

§173. Again, take the same quantities, and multiply their difference, by their difference.

$$\begin{array}{r} a-b \\ a-b \\ \hline a^2-ab \\ -ab+b^2 \\ \hline a^2-2ab+b^2 \end{array}$$

Therefore, *the square of the difference of two numbers, is equal to the square of the first number, minus twice the product of the two numbers, plus the square of the second.*

§174. The only difference between the square of the sum, and the square of the difference, is in the second term; being in one, positive, and in the other, negative. Let us subtract one from the other.

$$\begin{array}{c} a^2+2ab+b^2 \\ a^2-2ab+b^2 \\ \hline 4ab \end{array}$$

The actual difference between the square of the sum, and the square of the difference, is four times the product of the two numbers.

§175. If when the *sum* of two quantities has been raised to the second power and the co-efficient of the second term has been rejected, the quantity thus obtained be multiplied by the difference of the two original quantities; the result will be the *difference of the third powers of the two quantities.* Also, if we perform the same operation with the *difference* of the two quantities, multiplying by their sum, we shall obtain *the sum of the third powers of the two quantities.* Thus,

$$\begin{array}{ll} a+b & a-b \\ a+b & a-b \\ \hline a^2+2ab+b^2 & a^2-2ab+b^2 \end{array}$$

Rejecting the co-efficients,

$$\begin{array}{ll} a^2+ab+b^2 & a^2-ab+b^2 \\ a-b & a+b \\ \hline a^3+a^2b+ab^2 & a^3-a^2b+ab^2 \\ \quad -a^2b-ab^2-b^3 & \quad +a^2b-ab^2+b^3 \\ \hline a^3-b^3 & a^3+b^3 \end{array}$$

§176. It is often the case that it is better to *denote* the multiplication of compound quantities, than to perform it; on account of operations that follow. Thus, $a-b$ times $x+y$, may be written $(a-b).(x+y)$. No general rule can be given to determine when one method is preferable to the other. Experience is the best teacher in this particular. But it was thought best to mention it in this place.

When, after the multiplication has been denoted, the several terms are actually multiplied, the expression is said to be *expanded.*

DIVISION OF ALGEBRAIC QUANTITIES.

§177. Division may be *represented* by the sign $\div$, as $a \div b$, is read a, *divided by* b; $(a+b) \div (c-d)$, is read $a+b$, *divided by* $c-d$. But the most usual way to denote division (§67,) is to write the divisor underneath the dividend; thus, $\frac{a}{b}$, $\frac{a+b}{c-d}$.

§178. But it often happens (§71, 73,) that the fraction made by this representation is an improper fraction; and one in which the numerator can be actually divided by the denominator. In such cases, it is generally best to perfom the division. We have always done so in the former part of this treatise. Thus, $4x$ divided by $2=2x$; $\frac{10x}{5}=2x$.

§179. Let us first look at the case where the same quantity is in both the dividend and the divisor. $7a\div 7=a$; $12b\div 12=b$. In the same manner, $ab\div a=b$; $dc\div d=c$. This may be easily proved. For, ab is the product of a into b; and of course if we divide by what was the multiplier, we shall obtain the old multiplicand again; as may be seen by trying the product of any two numbers.

§180. Whence we derive the general rule, that *when the divisor is found as a factor in the dividend, the division is performed by erasing that factor from the dividend.* $amn\div a=mn$, because $amn=a$ times mn. $amn\div m=an$, because $amn=m$ times an. $amn\div n=am$, because amn is n times am.

EXAMPLES.

1. Divide $8c$ by 8. Ans. c.
2. Divide bc by b. Ans. c.
3. Divide $7m$ by 7. Ans. m.
4. Divide am by a. Ans. m.
5. Divide bd by b.
6. Divide $7ab$ by 7.
7. Divide cab by c.
8. Divide bad by b.
9. As abd is the same product as the last, divide that by b. Ans. ad.
10. Divide cde by c.
11. Divide the same product in another form, thus, dec by c. Ans. de.
12. Divide abc by c.
13. Divide abc by b.

§181. As $a^2=aa$, it is evident that if we divide a^2 by a the quotient is a; because we take away one of the written a's. So, if we divide a^5 by a, the quotient is a^4; because a^5 is the same as $aaaaa$, which $\div a$, gives $aaaa$ or a^4. So, if we divide a^5 by a^2, the quotient is aaa or a^3. And if we divide a^5 by a^4, the quotient is a; because $aaaaa \div aaaa = a$. Hence we see, that when *there are exponents in either the dividend or divisor, the division is performed by subtracting the exponent of the divisor from the exponent of the dividend.* $b^5 \div b^2 = b^3$; $x^7 \div x^3 = x^4$.

§182. As every literal quantity is understood to have the number 1 for its co-efficient, it is evident that if we divide by 1, the quotient would be the same literal quantity. Thus, $a \div 1 = a$. And again, if we divide a literal quantity by itself, the quotient will be 1. Thus, $1a \div a = 1$; the divisor being a factor in the dividend.

§183. It sometimes happens that the co-efficient contains the divisor as a factor. Thus, $8a$ is the same as 2 times $4a$, or 4 times $2a$; and therefore can be divided by 2 or by 4. In the same manner $8ab$ may be divided by $2a$, or $4a$, or $2b$, or $4b$; because it is $2a$ times $4b$, or $4a$ times $2b$.

We have only to remember to take those factors out of the dividend which are equal to the divisor. And in general, *when there are co-efficients in both the divisor and the dividend, divide the co-efficient of the dividend by the co-efficient of the divisor;* and then proceed with the literal quantities as before directed. $10abc \div 5b = 2ac$; $12a^3xy \div 2a^2y = 4ax$.

EXAMPLES.

14. Divide a^3 by a^2. Ans. a.
15. Divide x^6 by x^2. Ans. x^4.
16. Divide a^3b^4y by a^2b. Ans. ab^3y.
17. Divide $d^3c^4x^7$ by dc^2x^4. Ans. $d^2c^2x^3$.
18. Divide a^4m^5x by a^4m^2.
19. Divide $a^4x^6y^7$ by ax^4y^3.
20. Divide d^6y^7 by dy.
21. Divide $p^3r^7s^2t$ by r^5st.
22. Divide $ab^3c^4d^8$ by ab^2cd^5.
23. Divide ax^7y^8 by ax^3y^5.
24. Divide $c^5r^7s^3tx^4y^7$ by $c^2r^5ty^2$.
25. Divide $6a^2bc^7$ by $2bc^5$. Ans. $3a^2c^2$.
26. Divide $12ax^2y^4$ by $4ay^2$. Ans. $3x^2y^2$.
27. Divide $21bc^3xy^6$ by $3cy^2$.
28. Divide $42c^7d^8x^3$ by $6c^2d^5x$.
29. Divide $36pr^3st^4$ by $4rst^3$.

30. Divide $54m^5n^2x^3y$ by $9m^3ny$.

31. Divide $66a^2c^3dx^6$ by $3a^2c^5x^3$.

32. Divide $48c^2r^7x^3y^5$ by $8cr^5xy^2$.

33. Divide $72a^3r^3m^4$ by $18a^2rm^3$.

§184. We have shown, §162, that $(a+b)\times c=ac+bc$. Of course $(ac+bc)\div c=a+b$; where we see that *when we divide a compound quantity, we divide each of the terms.*

§185. We must also recollect that, as $+$ multiplied by $+$, makes $+$ in the product, so $+$ in the product divided by $+$, must make $+$ in the quotient. And that as $+$ multiplied by $-$ makes $-$, so $-$ in the product divided by $-$ will bring back the $+$ in the quotient. So that *when the signs are alike in the dividend and divisor, the sign in the quotient is* $+$. Thus, $a\times b=ab$; both of which are $+$. Also $-a\times -b=ab$; and $-ab\div -b=+a$.

§186. Again, as $-$ multiplied by $+$ makes $-$, so in the product $-$ divided by $+$, brings back $-$ in the quotient. Also, $-$ multiplied by $-$ makes $+$; and of course, $+$ in the product divided by $-$, brings $-$ in the quotient. That is, *when the signs in the divisor and dividend are unlike, the sign in the quotient is* $-$.

EXAMPLES.

34. Divide $2ad+8a^2c$ by $2a$. Ans. $d+4ac$.

35. Divide $8d^3m^2-12d^5m^3$ by $4dm^2$. Ans. $2d^2-3d^4m$.

36. Divide $4xy+6x^2$ by $2x$. Ans. $2y+3x$.

37. Divide $abc-acd$ by ac. Ans. $b-d$.

38. Divide $12ax-8ab$ by $-4a$. Ans. $-3x+2b$.

39. Divide $10xz+15xy$ by $5x$.

40. Divide $15ax-27x$ by $3x$.

41. Divide $18x^2-9x$ by $9x$.

42. Divide $abc-bcd-bcx$ by $-bc$.

43. Divide $3x+6x^2+3ax-15x$ by $3x$.

44. Divide $3abc+12abx-9a^2b$ by $3ab$.

45. Divide $40a^3b^2+60a^2b^2-17ab$ by ab.

46. Divide $15a^2bc-10acx^2+5ad^2c$, by $-5ac$.

47. Divide $20ax+15ax^2+10ax-5a$, by $5a$.

§187. It is evident that we may divide by either factor. Thus, $ax+bx$ may be divided by x, and the quotient will be $a+b$; or it may be divided by $a+b$, and the quotient will be x. This may appear singular to the young pupil; but he is to recollect that division is merely separating the dividend into factors, being careful to make one of them of a given magnitude; that is to make it the same as the given divisor.

§188. Now we know that x times $a+b,=ax+bx$; and also that $a+b$ times $x,=ax+bx$. Hence we know that the product $(ax+bx)\div(a+b)=x$. Therefore we conclude that *if the divisor contains just as many terms as the dividend, with corresponding signs; and the first term of it is a factor in the first term of the dividend, the second term of it in the second of the dividend, and so on through each of them respectively; and the remaining factor in each term of the dividend being the same; that remaining factor shall be the quotient.*

EXAMPLES.

48. Divide $ax+bx-cx$, by $a+b-c$. Ans. x.

49. Divide $bac+bc^2x-bx^3$, by $ac+c^2x-x^3$. Ans. b.

50. Divide $c^2ax-2abx-3xy+x$, by $ac^2-2ab-3y+1$.

51. Divide $cd^2x-abd^2+d^2x^3-d^2$, by $cx-ab+x^3-1$.

52. Divide $a^2y-bcy+xy$, by a^2-bc+x.

53. Divide $6ahm-14abm-3cdm$, by $6ah-14ab-3cd$.

§189. If the letters of the divisor are not found in the dividend, the division is expressed, as we have before shown, §177, by writing the divisor underneath the dividend, in the form of a vulgar fraction.

EXAMPLES.

54. Divide $4y+7x$, by $a-b$. Ans. $\frac{4y+7x}{a-b}$.

55. Divide $3a+2b^2-c$, by $a+c$.

56. Divide $a^3-x^2b+c^5$, by a^3-b^2.

57. Divide $3a^2c+2b^3+c$, by $2c$.

§190. When the dividend is a compound quantity, the divisor may be placed underneath the *whole* dividend if we choose. It may also be placed under *each term* of the dividend, which is the same as dividing each term, according to §184. By this method the answer of the last sum would be $\frac{3ac^2}{2c}+\frac{2b^3}{2c}+\frac{c}{2c}$. Answer the following by both methods.

58. Divide $3a+b+2ab$, by a. Ans. $\frac{3a}{a}+\frac{b}{a}+\frac{2ab}{a}$.

59. Divide $6a+ab-3b$, by $2b$.

60. Divide $2x+2y+3ax-2a^2y$, by $3ay$.

61. Divide $ax^2-bx-a^2b^2+ab$, by $2b$.

§191. When we divide each term separately, we may use both methods of division; that is, we may actually *divide* such terms as we can by §180; and merely *express* the division in such terms as cannot be divided.

EXAMPLES.

62. Divide $cd-ax+ac+bc$, by c. Ans. $d-\frac{ax}{c}+a+b$.

63. Divide $ax+bx-2ab+2x$, by x. Ans. $a+b-\frac{2ab}{x}+2$.

64. Divide $2am-3a^2b+b^3m-3a^2m$, by $-a$.

Ans. $-2m+3ab-\frac{b^3m}{a}+3am$.

65. Divide $2b^2-a^3b+3b^3c+a^3b^3-ac$, by b^2.

66. Divide $ay^3-by^2+4a^3b^2-5a^3by^2+aby$, by ay.

67. Divide $abx^3+a^3by-3ab^2x+ax^2-7a^2y$, by ab.

68. Divide $2abm+6a^2b+5b^2m-4a^3m$, by ab.

69. Divide $by-a^3b^2y+ay-aby+b^3y$, by $-by$.

70. Divide $ax-bx^3+xy^2+by-ay$, by $-y$.

In all of our divisions so far, it has been the case that either the divisor or the quotient is a simple quantity. For the method of dividing when *both of them are compound quantities*, see §213, &c.

EXERCISES IN EQUATIONS.

§192. It may be well now to generalize the questions in section 5 of Equations. And with this section, we will make our calculations more purely algebraical than in the preceding sections; as we shall take care to use *no numeral quantities at all in stating the questions.*

1. Two persons, A and B, lay out equal sums of money in trade. A gains c (\$126,) and B loses d (\$87;) and now A's money is m (two) times as much as B's. What did each lay out? See page 45.

Let $x=$ what each lay out.

$x+c=$ A's sum now.

$x-d=$ B's sum now.

$mx-md=$ m times B's.

Forming the equation, $mx-md=x+c$

Transposing, $mx-x=c+md$

Dividing by $m-1$, $x=\frac{c+md}{m-1}$.

Substituting numbers for letters, $\frac{c+md}{m-d}=\frac{126+2\times 87}{2-1}=300.$

In sums of this kind, the only difficulty is to determine what quantity to divide by in the last step, to leave x alone. But this difficulty is easily overcome, by dividing mentally the left hand member by x, and observing the quotient. Of course, dividing the same member by that *quotient* will produce x; which is our only object.

2. The 2d sum on page 45, is performed as follows:

$x=$ the wife's age.

$mx=$ the man's age.

$x+a=$ wife's after a years.

$mx+a=$ man's after a years.

$nx+na=$ n times the wife's age.

Forming the equation, $mx+a=nx+na$

Transposing terms, $mx-nx=na-a$

Dividing by $m-n$. $x=\dfrac{na-a}{m-n}$.

Substituting numbers, $x=15$, the age of his wife.

§193. It is customary to represent those numbers which stand for *times*, by the letters m, n, p, q, &c.

All the other questions in section 5th are to be performed in the foregoing manner.

FRACTIONS.

Reduction of Fractions to Lower Terms.

§194. We showed in §84, that a fraction may be reduced to lower terms, without any alteration in its value, by simply dividing both terms by a number that will divide each without a remainder. Fractions that are expressed by literal quantities may frequently be reduced in the same manner. Thus, in the fraction $\frac{axy}{ab}$, both the terms may be divided by a; and the fraction will then become $\frac{xy}{b}$.

EXAMPLES.

1. Reduce to the lowest terms the fraction $\frac{4abcx}{6adcy}$.

both terms $\left\{ \frac{4abcx}{6adcy} \right. \div 2ac = \frac{2bx}{3dy}$, Ans.

2. Reduce $\frac{a^2m^3y}{a^3bm^2x}$ to its lowest terms. Ans. $\frac{my}{abx}$.

3. Reduce $\frac{56x^4y^6}{-7x^3y^4}$ to its lowest terms. Ans. $-\frac{8y^2}{x^4}$.

4. Reduce $\frac{-4x^5y^2z}{5x^2y^4}$ to its lowest terms. Ans. $-\frac{4x^3z}{5y^2}$.

5. Reduce $\frac{-12x^4yz}{-4x^3yz^4}$ to its lowest terms. Ans. $\frac{3x}{z^3}$.

☞ These examples will remind the pupil, that, (because fractions are merely *expressions of division*,) when each term has its sign, then the whole fraction will have a sign according to §185 and 186.

6. Reduce $\frac{6a^2b^3x}{14ab^4x^2}$ to its lowest terms.

7. Reduce $\frac{9x^3y^5}{45ay^2}$ to its lowest terms.

8. Reduce $\frac{15a^3r^2s}{27amst}$ to its lowest terms.

§195. When we divide a compound quantity by a simple quantity we divide each term, §184. Hence, in reducing fractions to lower terms, we must find for the divisor, a quantity that is a factor in *every term* both of the numerator and denominator.

EXAMPLES.

9. Reduce $\frac{a^2x+ay^3}{a^3b}$ to its lowest terms. Ans. $\frac{ax+y^3}{a^2b}$.

10. Reduce $\frac{a^2x^3+ax^2-3a^4x}{ax^4-6ax^2+9a^2x}$ to its lowest terms.

Ans. $\frac{ax^2+ax-3a^3}{x^3-6x+9a}$.

11. Reduce $\frac{4xy-8x+12x^2y}{8ax^3-4a^2x}$ to its lowest terms.

12. Reduce $\frac{3a^3b^2x-9b^2x-6ax^2}{12ax+6abx-15bx}$ to its lowest terms.

13. Reduce $\frac{7a^2r^3m-56am-14am^2}{35a^2m+21abm+56am}$ to its lowest terms.

14. Reduce $\frac{33a^2xy^3+18ax^2y-6ay}{3a^4y^4-21a^3y^3-15a^2y^2-9ay}$ to its lowest terms.

15. Reduce $\frac{112abcx-48acd+100acx}{4abcd-8acd+52acx}$ to its lowest terms.

§196. By this principle, we may often simplify answers to questions in division. That is, we may put the divisor

underneath the dividend, so as to make a fraction, and then reduce that fraction to its lowest terms.

EXAMPLES.

16. Divide $x^2-2xy+xy^2$ by $4xy$. Ans. $\frac{x-2y+y^2}{4y}$.

17. Divide $10xy-20x-5y$ by $-5x$. Ans. $-\frac{2xy-4x-y}{x}$.

18. Divide $7abx-56a^2xy+14ax^3$ by $28a^3bx^2y$.

19. Divide $8amy^3+16a^3xy-24aby^2$ by $48a^2y^2-72ay$.

20. Divide $35a^3bc-14ax+42a^3$ by $21a^6-28a^5x+7a^4x^2$.

21. Divide $32x^3y+16x^2y^2+8xy^3$ by $24ax+48a^2x^2$.

22. Divide $54a^2b^2+45a^3b^2-27a^2b^3$ by $24a^5b^2+30a^3b^5$.

23. Divide $48x^2y^3+12axy-16ax$ by $4x^2y^2+6ax-8xy$.

24. Divide $28(a-b+x)$ by $4m(a-b+x.)$ Ans. $\frac{7}{m}$.

25. Divide $12cd(m-n)$ by $14ac(m-n)$.

26. Divide $6ah(r+p)$ by $24a(r+p)$.

27. Divide $(a+b)(m+n)$ by $(a-x)(m+n)$.

§197. It must be remembered that when all the factors in the numerator are contained in the denominator, the answer will contain 1 in the numerator. Thus, $\frac{3a}{3ax}=\frac{1}{x}$.

MULTIPLICATION WHERE ONE FACTOR IS A FRACTION.

§198. This is done, (as shown §69, 70,) by multiplying the whole number and the numerator of the fraction together, and dividing by the denominator. Thus, $2a\times\frac{3b-x}{y}$ $=\frac{6ab-2ax}{y}$.

EXAMPLES.

1. Multiply $\frac{xy}{3a}$ by $3a$. Ans. $\frac{3axy}{3a}=xy$.

2. Multiply $\frac{3z^2}{8ax}$ by $8a$. Ans. $\frac{3z^2}{x}$.

3. Multiply $3ax$ into $-\frac{ab}{12ax}$. Ans. $-\frac{ab}{4}$.

4. Multiply $2ab-3xy$ by $\frac{2by}{3ax}$. Ans. $\frac{4ab^2y-6bxy^2}{3ax}$.

5. Multiply $\frac{aby}{cx}$ by $2ax-3xy$. Ans. $\frac{2a^2by-3aby^2}{c}$.

6. Multiply $3am^2-4x$ by $\frac{3x^2+a}{2m}$.

7. Multiply $\frac{5ast=2m}{4ax}$ by $am-2a^2$.

8. Multiply $3a^2y+6x^3$ by $\frac{2ab+cd}{3ax-bd}$.

9. Multiply $\frac{2ar^3}{3ax-bx}$ by $2ax+3x^2$.

10. Multiply $14abc-3cdx$ by $\frac{2am}{5mx-2cm}$.

11. Multiply $\frac{10ax-6xy}{14am}$ by $3mx-2am$.

12. Multiply $\frac{a}{b}$ by b. Ans. $\frac{ab}{b}=a$.

13. Multiply $\frac{ab}{cd}$ by c. Ans. $\frac{abc}{cd}=\frac{ab}{d}$.

§199. In the last example, we first multiplied the numerator by c and then divided both the numerator and the denominator by c. Now, multiplying the numerator by c

and then dividing it by c, is altogether useless; because the numerator is left as it was at first. We will use then only one part of the operation; that is, dividing the denominator by c. And in general, *when the multiplier is a factor in the denominator, the multiplication is performed by canceling that factor.*

EXAMPLES.

14. Multiply $\frac{amx}{rs}$ by r. Ans. $\frac{amx}{s}$.

15. Multiply $\frac{abc}{x^3y}$ by x^2. Ans. $\frac{abc}{xy}$.

16. Multiply $\frac{ax^3y}{2cdn}$ by $2cn$. Ans. $\frac{ax^3y}{d}$.

17. Multiply $\frac{3ars}{2cdn}$ by $2d$.

18. Multiply $\frac{ab+ax}{2bx}$ by $2x$.

19. Multiply $\frac{ax^3+2a}{3abx}$ by $3bx$.

20. Multiply $\frac{am^3}{12ax^3-18bx^2}$ by $6x^2$.

21. Multiply $\frac{6am-4x}{3am+6ax-12a^3}$ by $3a$.

22. Multiply $\frac{4bc-bx+cx}{9axy+12a^3x-6axc}$ by $3ax$.

23. Multiply $\frac{am+4ab-2m}{8abm+16ab-4abx}$ by $4ab$.

DIVISION OF FRACTIONS.

§200. We have shown in §95, that *we divide fractions by dividing the numerator, when the divisor is a factor in*

the numerator ; but if the divisor is not a factor in the numerator, we multiply the denominator by the divisor.

EXAMPLES.

1. Divide $\frac{3ab}{x}$ by a. Ans. $\frac{3b}{x}$.

2. Divide $\frac{8ax}{bc}$ by b. Ans. $\frac{8ax}{b^2c}$.

3. Divide $\frac{3xy-6x}{2b}$ by $3x$. Ans. $\frac{y-2}{2b}$.

4. Divide $\frac{xy}{z}$ by $-3a$. Ans. $-\frac{xy}{3az}$.

5. Divide $\frac{xyz}{4}$ by $7a^2$.

6. Divide $\frac{60xy}{4z}$ by 7.

7. Divide $\frac{3ax-6ay}{4axy}$ by $2xy$.

8. Divide $\frac{3ax-y}{a+y}$ by $3ay$.

9. Divide $\frac{4a+27ab}{2x}$ by $6ax$.

10. Divide $\frac{6abx-16axy}{2am+3xy}$ by $2a$.

11. Divide $\frac{14bc^2x+21ac^3}{4abx}$ by $7c^2$.

12. Divide $\frac{10a^5+5x^5-15ax}{a^4+5x^2-10ax}$ by $5a^5x^4$.

13. Divide $\frac{12ab-14cx}{10ax+16bc}$ by $2ac-4bx$.

14. Divide $\frac{16cd+8x}{x-3y}$ by $4cd-cax$.

15. Divide $\frac{9am+12ax}{6mx}$ by $12ax-9a^2$.

16. Divide $\frac{9a^2+21}{14a^3-16}$ by a^2-1.

EXERCISES IN EQUATIONS.

§201. Generalize the questions in Section 6th, page 54.

1. In an orchard, $\frac{1}{m}$ $(\frac{1}{4})$ of the trees bear apples; $\frac{1}{n}$ $(\frac{1}{5})$ of them bear pears; $\frac{p}{r}$ $(\frac{2}{11})$ of them plums, and a (81) bear cherries. How many trees are there in the orchard?

☞ We rarely represent *unity* by a letter; but generally use its own character, 1.

Let $x =$ number of trees.

$\frac{x}{m} =$ apple-trees.

$\frac{x}{n} =$ pear-trees.

$\frac{px}{r} =$ plum-trees.

Then, $$x=\frac{x}{m}+\frac{x}{n}+\frac{px}{r}+a.$$

Multiplying by mnr, $mnrx=nrx+mrx+mnpx+mnra$

Transposing, $mnrx-nrx-mrx-mnpx=mnra$

Dividing by $mnr-nr-mr-mnp$,

$$x=\frac{mnra}{mnr-nr-mr-mnp}.$$

Substitute figures for letters, and find the answer.

2d sum, page 55. In a certain school, $\frac{1}{m}$ of the boys learn mathematics; $\frac{p}{n}$ of them study Latin and Greek; and a study Grammar. What is the whole number of scholars?

The equation is, $x = \frac{x}{m} + \frac{px}{n} + a.$

Multiplying by mn, $mnx = nx + mpx + mna.$

Transposing, $mnx - nx - mpx = mna.$

Dividing by $mn - n - mp$, $x = \frac{mna}{mn - n - mp}.$

3. The pupil may go through with the whole section in the same manner. And as there are several instances in which the same statement and the same answer will agree with two or more sums, the pupil may tell which they are, and why it so happens.

§202. If the teacher should think his pupils need more practice, he may exercise them in the 7th section in the same manner.

The first question may be stated thus,

$$\frac{x + a}{b} = c$$

Multiplying by b, $x + a = bc$

Transposing, $x = bc - a.$

The 2d will have the following equation.

$$\frac{mx + a}{n} = b$$

Multiplying by n, $mx + a = nb$

Transposing and dividing, $x = \frac{nb - a}{m}.$

The 3d and 5th will have the following equation.

$$\frac{nx - na}{m} = b.$$

The 4th will be, $a - x = \frac{nx}{m}.$

FRACTIONS OF FRACTIONS.

§203. It was shown, §96, *that a fraction is multiplied by a fraction, by multiplying the numerators together for a new numerator, and the denominators together for a new denominator*. Thus, $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

EXAMPLES.

1. Multiply $\frac{3a}{5x}$ by $\frac{4b}{c}$. Ans. $\frac{12ab}{5cx}$.

2. Multiply $\frac{9ax}{5b}$ by $\frac{2bx}{3ay}$. Ans. $\frac{18abx^2}{15aby}=\frac{6x^2}{5y}$.

3. Find the product of $\frac{x+y}{3a}$ into $\frac{3y}{x-y}$. Ans. $\frac{xy+y^2}{ax-ay}$.

4. Multiply $\frac{3x}{4}$ and $\frac{4x}{5}$ together. Ans. $\frac{3x^2}{5}$.

5. Multiply $\frac{x}{3y}$ into $\frac{6y}{7x}$. Ans. $\frac{2}{7}$.

6. Multiply $\frac{3a}{4b}$ into $-\frac{4a}{5}$. Ans. $-\frac{3a^2}{5b}$.

7. Multiply $\frac{x}{a}$ into $\frac{x+y}{x+2y}$. Ans. $\frac{x^2+xy}{ax+2ay}$.

8. Multiply $-\frac{a}{x}$ into $-\frac{y}{z}$. Ans. $\frac{ay}{xz}$.

9. Multiply $\frac{a}{x}$, $\frac{3x}{y}$, $\frac{4y}{3z}$ together. Ans. $\frac{4a}{z}$.

10. What is the product of $\frac{4ax}{y}$, $\frac{3xy}{2a}$, and $\frac{2}{x}$? Ans. $12x$.

11. What is the product of $\frac{2a}{3b+c}$ into $\frac{2ac-bc}{5ab}$?

12. Multiply $\frac{2am^2 - 3a^2m}{4ac + 2c}$ by $\frac{5am^2}{2am - 5c}$.

13. Multiply $\frac{10xy^2 + 5x^2y}{4x - 3y}$ by $\frac{2x^2y - xy^2}{5xy^2 + 10x^2y}$.

EXERCISES IN EQUATIONS.

§204. Generalize the questions in Equations, Section 8.

1. The 1st sum on page 66, is performed as follows.

Stating the question, $x =$ oats.

$$\frac{mx}{n} = \text{barley}.$$

$$\frac{1}{p} \text{ of } \frac{mx}{n} = \frac{mx}{np} = \text{rye}.$$

Forming the equation, $x + \frac{mx}{n} + \frac{mx}{np} = a$

Multiplying by np, $npx + mpx + mx = anp$

Dividing by $np + mp + m$, $x = \frac{anp}{np + mp + m}$.

2. The equation in the 2d sum will be $\frac{x}{n} - \frac{x}{mn} = a$

Multiplying by mn, $mx - x = amn$

Dividing by $m - 1$, $x = \frac{amn}{m - 1}$.

3. The equation in the 3d sum will be

$$x - \frac{x}{m} - \frac{x}{n} + \frac{x}{mn} = a.$$

4. The equation in the 4th sum will be

$$x - \frac{nx}{m} - \frac{nx}{m} + \frac{n^2x}{m^2} = a.$$

The pupil will now understand how to perform the rest of the section.

UNITING FRACTIONS OF DIFFERENT DENOMINATIONS.

§205. Before fractional terms can be united, they must be brought to a common denominator, according to the principle explained in §81. *This is done* as we have shown in §102, 103, *by multiplying each numerator by all the denominators except its own, for new numerators; and all the denominators together for a new denominator.*

EXAMPLES.

1. Unite the following terms, $\frac{2ab}{x}-\frac{ax}{4b}+\frac{am}{bx}$.

Ans. $\frac{8ab^3x-abx^3+4abmx}{4b^2x^2}=\frac{8ab^2-ax^2+4am}{4bx}$.

2. Add $\frac{x}{x+1}$ and $\frac{x}{x-1}$ together. Ans. $\frac{2x^2}{x^2+1}$.

3. Subtract $\frac{3a}{8c}$ from $\frac{4a}{7b}$. Ans. $\frac{32ac-21ab}{56bc}$.

4. Unite $\frac{x}{x+y}-\frac{y}{z}$. Ans. $\frac{xz-xy-y^2}{xz+yz}$.

5. Unite $\frac{x-y}{2a}-\frac{x+y}{3a}$. Ans. $\frac{ax-5ay}{6a^2}$.

§206. If, in such cases, there is a quantity that is not fractional, we multiply it by all the denominators; and then putting the common denominator under that product, unite it with the other fractions.

6. Unite $a+\frac{b}{c}$. Ans. $a=\frac{ca}{c}$; and then, $\frac{ca}{c}+\frac{b}{c}=\frac{ac+b}{c}$.

7. Unite $x+\frac{a+x}{y}$. Ans. $\frac{xy+a+x}{y}$.

8. Unite $a-\frac{a+b}{c}$. Ans. $\frac{ac-a-b}{c}$.

9. Unite $\frac{a}{2x}+\frac{4}{y}-\frac{x}{z}$. Ans. $\frac{ayz+8xz-2x^2y}{2xyz}$.

10. Unite $\frac{x+y}{z}+\frac{a-b}{c}$. Ans. $\frac{cx+cy+az-bz}{cz}$.

11. Unite $\frac{a}{a+2}+\frac{3ax}{2y^2}$. Ans. $\frac{2ay^2+3a^2x+6ax}{2ay^2+4y^2}$.

12. Unite $x+\frac{a}{b}-ay-\frac{4}{7}$. Ans. $\frac{7bx+7a-7aby-4b}{7b}$.

13. Add $\frac{2x+1}{3}$, $\frac{4x+2}{5}$, and $\frac{x}{7}$ together. Ans. $\frac{169x+77}{105}$.

14. Add together $\frac{5a^2+b}{3b}$, and $\frac{4a^2+2b}{5b}$. Ans. $\frac{37a^2+11b}{15b}$.

15. Subtract $\frac{5x+1}{7}$ from $\frac{21x+3}{4}$. Ans. $\frac{127x+17}{28}$.

16. Subtract $\frac{3x+1}{x+1}$ from $\frac{4x}{5}$. Ans. $\frac{4x^2-11x-5}{5x+5}$.

17. Add together $\frac{2x-5}{3}$, and $\frac{x-1}{2x}$. Ans. $\frac{4x^2-7x-3}{6x}$.

18. Add together $\frac{x}{x-3}$, and $\frac{x}{x+3}$. Ans. $\frac{2x^2}{x^2-9}$.

19. Subtract $\frac{2x-3}{3x}$ from $\frac{4x+2}{3}$. Ans. $\frac{4x^2+3}{3x}$.

20. Add together $\frac{a+b}{a-b}$, and $\frac{a-b}{a+b}$. Ans. $\frac{2a^2+2b^2}{a^2-b^2}$.

21. From $\frac{3a+2b}{c}$, subtract $\frac{5bd-2a-3d}{4cd}$.

Ans. $\frac{12ad+3bd+2a+3d}{4cd}$.

22. From $c+2ab-3ac$, subtract $\frac{b^2c-5ab^2c+a^3}{b^2-bc}$.

Ans. $\frac{2ab^3-bc^2+3abc^2-a^3}{b^2-bc}$.

DIVISION BY FRACTIONS.

§207. Suppose we wish to know how many times $\frac{3}{7}$ is contained in $\frac{6}{7}$. We would divide in the same manner that we follow in dividing 6 pieces by 3 pieces; and say $\frac{3}{7}$ is contained in $\frac{6}{7}$, *two* times. In the same manner, $\frac{4}{21}$ is contained in $\frac{20}{21}$, *five* times.

The principle is general that *when the divisor and the dividend have a common denominator, the division is performed by dividing the numerator of the dividend by the numerator of the divisor.*

$$\text{Thus, } \frac{a}{b} \div \frac{c}{b} = \frac{a}{c}; \quad \frac{a^3}{x} \div \frac{a^2}{x} = \frac{a^3}{a^2} = a.$$

EXAMPLES.

1. Divide $\frac{4a}{x}$ by $\frac{2b}{x}$. Ans. $\frac{4a}{2b} = \frac{2a}{b}$.

2. Divide $\frac{3ab}{cd}$ by $\frac{4bx}{cd}$. Ans. $\frac{3ab}{4bx} = \frac{3a}{4x}$.

3. Divide $\frac{2x}{ab}$ by $\frac{3xy}{ab}$. Ans. $\frac{2}{3y}$.

4. Divide $\frac{3a-b}{ab}$ by $\frac{2a^2}{ab}$. Ans. $\frac{3a-b}{2a^2}$.

5. Divide $\frac{7rx+a^2}{5ast}$ by $\frac{4b+ax}{5ast}$. Ans. $\frac{7rx+a^2}{4b+ax}$.

6. Divide $8a$ by $\frac{2ab}{xy}$.

Explanation. By §206, $8a = \frac{8axy}{xy}$. Then $\frac{8axy}{xy} \div \frac{2ab}{xy} = \frac{8axy}{2ab} = \frac{4xy}{b}$

7. Divide $3a$ by $\frac{4x}{y}$. Ans. $\frac{3ay}{4x}$.

8. Divide ab by $\frac{c}{d}$. Ans. $\frac{abd}{c}$.

9. Divide $\frac{4a^2x}{y}$ by $\frac{2x^3}{y}$. Ans. $\frac{2a^2}{x^2}$.

10. Divide $\frac{8mnr}{ax}$ by $\frac{6am^2}{ax}$. Ans. $\frac{4nr}{3am}$.

11. Divide $\frac{4ay^2x}{bc}$ by $\frac{10ax^2}{bc}$. Ans. $\frac{2y^2}{5x}$.

12. Divide $\frac{a+bc}{ax}$ by $\frac{4a^2-ab}{ax}$. Ans. $\frac{a+bc}{4a^2-ab}$.

13. Divide $\frac{am^2}{ab-c}$ by $\frac{ax}{ab-c}$. Ans. $\frac{m^2}{x}$.

14. Divide $\frac{ax-y}{ab}$ by $\frac{xy}{b}$.

Explanation. In this example, the dividend and the divisor has not a common denominator. Our first object then, is to bring them to a common denominator. This is done by §205, $\frac{ax-y}{ab} \div \frac{xy}{b} = \frac{bax-by}{ab^2} \div \frac{abxy}{ab^2}$.

Ans. $\frac{abx-by}{abxy} = \frac{ax-y}{axy}$.

15. Divide $\frac{abx}{c}$ by $\frac{dy}{a}$.

Operation. $\frac{abx}{c} \div \frac{dy}{a} = \frac{a^2bx}{ac} \div \frac{cdy}{ac} = \frac{a^2bx}{cdy}$.

§208. It will be seen that in sums of this kind, after we have brought the terms to a common denominator, the division is performed by putting the numerator of the dividend for the numerator of the answer, and the numerator of the

divisor for the denominator of the answer, and make no use at all of the denominators. Let us see then how we obtain these two terms. We multiply the numerator of the dividend by the denominator of the divisor; and this becomes the *numerator of the answer*. And we multiply the numerator of the divisor by the denominator of the dividend; and this becomes the *denominator of the answer*. By looking at the last two sums, it will be seen that this is the true operation.

§209. Hence we obtain the general *rule* for dividing by a fraction. *Multiply the numerator of the dividend by the denominator of the divisor, for a new numerator; and multiply the denominator of the dividend by the numerator of the divisor for a new denominator.*

§210. When the dividend is a whole number, it is changed into a fraction, by putting 1 under it for a denominator.

$$\text{Thus, } x = \frac{x}{1}; \qquad 2a = \frac{2a}{1}.$$

EXAMPLES.

16. Divide $\frac{3x}{4}$ by $\frac{ax}{2b}$. Ans. $\frac{6bx}{4ax} = \frac{3b}{2a}$.

17. Divide $\frac{a+x}{a-y}$ by $\frac{a+y}{a+2x}$. Ans. $\frac{a^2+3ax+2x^2}{a^2-y^2}$.

18. Divide $x^2-2ax+a^2$ by $\frac{1}{x-a}$.

Ans. $x^3-3x^2a+3a^2x-a^3$.

19. Divide $\frac{a}{4}$ by $\frac{3a}{5}$. Ans. $\frac{5}{12}$.

20. Divide $\frac{x-1}{3}$ by $\frac{x+1}{4}$. Ans. $\frac{4x-4}{3x+3}$.

21. Divide $\frac{a+x}{y}$ by $\frac{5a}{4b}$. Ans. $\frac{4ab+4bx}{5ay}$.

22. Divide $\frac{5ab}{4}$ by $-\frac{4a}{y}$ Ans. $-\frac{5by}{16}$.

23. Divide $\frac{a}{a+1}$ by $\frac{3x}{4y}$. Ans. $\frac{4ay}{3ax+3x}$.

24. Divide $x+ax$ by $\frac{3a}{x-y}$. Ans. $\frac{x^2+ax^2-xy-axy}{3a}$.

25. Divide $\frac{y}{y-1}$ by $\frac{y}{3}$. Ans. $-\frac{3}{y-1}$.

26. Divide $4a-ay$ by $\frac{4y-ay}{4a}$. Ans. $\frac{16a^2-4a^2y}{4y-ay}$.

27. Divide $\frac{1}{x-y}$ by $\frac{1}{x^2-y^2}$. Ans. $x+y$.

REDUCTION OF COMPLEX FRACTIONS TO SIMPLE ONES.

§211. We have shown, that when we multiply a fraction by its denominator, we obtain for the answer, the same quantity as the numerator. We have also shown that where both terms of a fraction are multiplied by the same quantity, the *value* of the fraction is not altered. By these two principles, we obtain the following *rule* for reducing a complex fraction to a simple one.

§212. *Multiply both terms of the fraction by the denominator that is found either in the entire numerator or denominator. If the fraction is still complex, multiply the result in both terms by the remaining denominator that is found in the entire term.* Thus,

$$\frac{a+\frac{b}{c}}{d}=\frac{ca+b}{cd}; \qquad \frac{a-\frac{b}{c}}{\frac{3}{4}+a}=\frac{ca-b}{\frac{3c}{4}+ca}=\frac{4ac-4b}{3c+4ac}$$

EXAMPLES.

1. Reduce $\frac{a}{a-\frac{b}{c}}$ to a simple fraction.

Ans. Multiplying both terms by c, $\frac{ac}{ac-b}$.

2. Reduce $\frac{a-\frac{x}{y}}{ax}$ to a simple fraction. Ans. $\frac{ay-x}{axy}$.

3. Reduce $\frac{x}{ay+\frac{y-z}{5}}$ to a simple fraction.

Ans. $\frac{5x}{5ay+y-z}$.

4. Reduce $\frac{x-\frac{x}{4}}{xy}$ to a simple fraction. Ans. $\frac{4x-x}{4yx}=\frac{3}{4y}$.

5. Reduce $\frac{2xy}{3a-\frac{y-1}{y}}$ to a simple fraction. Ans. $\frac{2xy^2}{3ay-y+1}$.

6. Reduce $\frac{x+\frac{x}{y}}{z-\frac{y}{c}}$ to a simple fraction.

Ans. $\frac{x+\frac{x}{y}}{z-\frac{y}{c}}=\frac{xy+x}{\frac{yz-y^2}{c}}=\frac{cxy+cx}{cyz-y^2}$.

7. Reduce $\frac{4+\frac{x}{z}}{x-\frac{5}{6}}$ to a simple fraction. Ans. $\frac{24z+6x}{6xz-5z}$.

8. Reduce $\frac{xy+\frac{x}{y}}{a+\frac{bx}{c}}$ to a simple fraction. Ans. $\frac{cxy^2+cx}{acy+bxy}$.

9. Reduce $\dfrac{3+\frac{4}{x}}{4-\frac{5}{y}}$ to a simple fraction. Ans. $\dfrac{3xy+4y}{4xy-5x}$.

10. Reduce $\dfrac{x-\frac{ay}{b}}{y+\frac{ax}{c}}$ to a simple fraction. Ans. $\dfrac{bcx-acy}{bcy+abx}$.

§213. It sometimes happens that we wish to transfer a fraction from a numerator to the denominator, or from a denominator to the numerator. This may be done by the foregoing principles. For, supposing we have $\dfrac{\frac{2}{7}a}{x}$; multiplying by the denominator of $\frac{2}{7}$, we have $\dfrac{2a}{7x}$. Now, if we divide this fraction by 2, we have $\dfrac{a}{\frac{7}{2}x}$. Thus we see the fraction is transferred, without altering the value of the whole quantity, if we take care to *invert* it when we transfer it.

THE END.